Solutions for *Active Portfolio Management* by Grinold and Kahn

Nicholas J. Hestand & Nicholas J. Williams

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Chapter 2

Problem 2.1. In December 1992, Sears had a predicted beta of 1.05 with respect to the S&P 500 index. If the S&P 500 Index subsequently underperformed Treasury bills by 5.0 percent, what would be the expected excess return to sears?

Solution. The excess return on the market (relative to the risk free asset Treasury bills) is -5%. Hence, the excess return to Sears is

$$\begin{aligned} r_{Sears} &= \beta_{Sears} r_M \\ &= 1.05 \times -5.0\% \\ &= -5.25\% \end{aligned}$$

Problem 2.2. If the long-term expected excess return to the S&P 500 Index is 7 percent per year, what is the expected excess return to Sears.

Solution. Using the same line of reasoning as above, we have

$$\begin{aligned} r_{Sears} &= \beta_{Sears} r_M \\ &= 1.05 \times 7.0\% \\ &= 7.35\% \end{aligned}$$

Problem 2.3. Assume that residual returns are uncorrelated across stocks. Stock A has a beta of 1.15 and a volatility of 35 percent. Stock B has a beta of 0.95 and a volatility of 33 percent. If the market volatility is 20 percent, what is the correlation of stock A with stock B? Which stock has higher residual volatility?

Solution. The variance of a portfolio P is given by eq. (2.4) as

$$\sigma_P^2 = \beta_P^2 \sigma_M^2 + \omega_P^2$$

where ω_P^2 is the residual variance and σ_M^2 is the market variance. The correlation of stock A with stock B is given by

$$\text{Corr}\{r_A, r_B\} = \frac{\text{Cov}\{r_A, r_B\}}{\text{Std}\{r_A\}\,\text{Std}\{r_B\}}$$

So we just need the covariance of stocks A and B. We can write

$$\text{Cov}\{r_A, r_B\} = \beta_A \beta_B \sigma_M^2 + \omega_{A,B}$$

where the cross terms have been omitted since the residual volatility is uncorrelated from the market volatility. We can also set the residual covariance ($\omega_{A,B}$) to zero since we are assuming that the residual returns are uncorrelated across stocks. Hence the correlation of stock A with stock B is

$$\begin{aligned}\text{Corr}\{r_A, r_B\} &= \frac{\text{Cov}\{r_A, r_B\}}{\text{Std}\{r_A\}\,\text{Std}\{r_B\}} \\ &= \frac{\beta_A \beta_B \sigma_M^2}{\sigma_A \sigma_B} \\ &= \frac{1.15 \times 0.95 \times (20\%)^2}{35\% \times 33\%} \\ &= 0.3784\end{aligned}$$

We can determine the residual volatility of stock P from

$$\omega_P = \sqrt{\sigma_P^2 - \beta_P^2 \sigma_M^2}$$

Hence,

$$\begin{aligned}\omega_A &= \sqrt{\sigma_A^2 - \beta_A^2 \sigma_M^2} \\ &= \sqrt{(35\%)^2 - 1.15^2 \times (20\%)^2} \\ &= 26.38\%\end{aligned}$$

$$\begin{aligned}\omega_B &= \sqrt{\sigma_B^2 - \beta_B^2 \sigma_M^2} \\ &= \sqrt{(33\%)^2 - 0.95^2 \times (20\%)^2} \\ &= 26.98\%\end{aligned}$$

so portfolio B has higher residual volatility.

Problem 2.4. What set of expected returns would lead us to invest 100 percent in GE stock?

Solution. According to the CAPM, investing in anything other than the market portfolio involves taking on excess risk. Hence, investing 100 percent in GE stock would expose us to unnecessary risk. In order to minimize risk, we should simply invest in the market portfolio. If we didn't care about risk, we would invest 100 percent in GE whenever the expected returns on the market are positive since GE has a historical beta of 1.3 (table 2.1), which is the highest beta of the MMI stocks in table 2.1.

Problem 2.5. According to the CAPM, what is the expected residual return of an active manager?

Solution. The CAPM states that the expected residual return on all stocks is zero.

Chapter 2 technical appendix

Problem 2a.1. Show that $\beta_C = \frac{\sigma_C^2}{\sigma_M^2}$. Since portfolio C is the minimum-variance portfolio, this relationship implies that $\beta_C \leqslant 1$, with $\beta_C = 1$ only if the market is the minimum-variance portfolio.

Solution. Since β_C is defined relative to the market portfolio, we have that the characteristic of the market portfolio is β by (2A.19). Hence β_C is equal to the exposure of portfolio C to the characteristic of the market portfolio. Then, by (2A.4), we have

$$e_M \sigma_C^2 = \beta_C \sigma_M^2.$$

We must have $e_M = 1$, since the balance of borrowing and lending in the market portfolio must balance out. Hence

$$\beta_C = \frac{\sigma_C^2}{\sigma_M^2}.$$

Problem 2a.2. Show that $f_Q = f_C + \frac{\sigma_C^2}{\kappa f_C}$, i.e., $\kappa = \frac{\sigma_C^2}{f_C(f_Q - f_C)}$

Solution. We begin with (2A.35):

$$\frac{f_C}{\sigma_C^2} = \frac{f_Q}{\sigma_Q^2}.$$

We then deduce

$$\begin{aligned} & f_C \sigma_Q^2 = f_Q \sigma_C^2 \\ \therefore \quad & f_C(\sigma_Q^2 - \sigma_C^2) = \sigma_C^2 (f_Q - f_C) \\ \therefore \quad & \frac{(\sigma_Q^2 - \sigma_C^2)}{(f_Q - f_C)^2} = \frac{\sigma_C^2}{f_C(f_Q - f_C)}. \end{aligned}$$

This then gives us that

$$\kappa = \frac{\sigma_C^2}{f_C(f_Q - f_C)},$$

as desired.

Problem 2a.3. What is the "characteristic" associated with the MMI portfolio? How would you find it?

Solution. By Proposition 1.3, the characteristic associated with a portfolio is the vector of betas of all assets with respect to the portfolio. This then applies to the MMI portfolio.

To find the characteristic, one could calculate these betas using regression.

Problem 2a.4. Prove that the fully invested portfolio that maximizes $f_P - \lambda \sigma_P^2$ has expected excess return $f^* = f_C + \frac{1}{2\lambda\kappa}$.

Solution. To start, we note the following facts, which we will need to use in this question, as well as later questions.

$$\mathbf{h}_C = \frac{\mathbf{V}^{-1}\mathbf{e}}{\mathbf{e}^T\mathbf{V}^{-1}\mathbf{e}}, \qquad (2A.14)$$

$$\sigma_C^2 = \frac{1}{\mathbf{e}^T\mathbf{V}^{-1}\mathbf{e}}, \qquad (2A.15).$$

We also have

$$\mathbf{h}_Q = \frac{\mathbf{h}_q}{e_q} = \frac{\mathbf{V}^{-1}\mathbf{f}}{\mathbf{f}^T\mathbf{V}^{-1}\mathbf{f}} \frac{\mathbf{f}^T\mathbf{V}^{-1}\mathbf{f}}{\mathbf{e}^T\mathbf{V}^{-1}\mathbf{f}} = \frac{\mathbf{V}^{-1}\mathbf{f}}{\mathbf{e}^T\mathbf{V}^{-1}\mathbf{f}},$$

where we draw upon (2A.23) and Proposition 3. We also note

$$\sigma_Q^2 = \frac{f_Q \sigma_C^2}{f_C} = \frac{\mathbf{f}^T\mathbf{V}^{-1}\mathbf{f}}{\mathbf{e}^T\mathbf{V}^{-1}\mathbf{f}} \frac{1}{\mathbf{e}^T\mathbf{V}^{-1}\mathbf{e}} \frac{\mathbf{e}^T\mathbf{V}^{-1}\mathbf{e}}{\mathbf{f}^T\mathbf{V}^{-1}\mathbf{e}} = \frac{\mathbf{f}^T\mathbf{V}^{-1}\mathbf{f}}{(\mathbf{e}^T\mathbf{V}^{-1}\mathbf{f})^2},$$

drawing upon (2A.35). Finally, note that $\mathbf{f}^T\mathbf{V}^{-1}\mathbf{e} = \mathbf{e}^T\mathbf{V}^{-1}\mathbf{f}$ because $\mathbf{V}$ is a symmetric matrix, and so $\mathbf{V}^{-1}$ is symmetric too.

In this question we need to maximise

$$\mathbf{h}^T\mathbf{f} - \lambda\mathbf{h}^T\mathbf{V}\mathbf{h}$$

subject to the constraint

$$\mathbf{e}^T\mathbf{h} = 1.$$

We use the method of Lagrange multipliers, considering the function

$$\mathbf{h}^T\mathbf{f} - \lambda\mathbf{h}^T\mathbf{V}\mathbf{h} - \theta(\mathbf{e}^T\mathbf{h} - 1)$$

and obtaining the equations

$$\begin{aligned}\mathbf{f} - 2\lambda\mathbf{V}\mathbf{h} - \theta\mathbf{e} &= 0,\\ \mathbf{e}^T\mathbf{h} &= 1.\end{aligned}$$

By rearranging the first equation, we obtain that

$$\mathbf{h} = \frac{1}{2\lambda}\mathbf{V}^{-1}(\mathbf{f} - \theta\mathbf{e}).$$

We then substitute this into the second equation, which gives us that

$$\frac{1}{2\lambda}\mathbf{e}^T\mathbf{V}^{-1}(\mathbf{f} - \theta\mathbf{e}) = 1.$$

We can now solve for θ:

$$\begin{aligned}\mathbf{e}^T\mathbf{V}^{-1}(\mathbf{f} - \theta\mathbf{e}) &= 2\lambda\\ \therefore\quad \theta\mathbf{e}^T\mathbf{V}^{-1}\mathbf{e} &= \mathbf{e}^T\mathbf{V}^{-1}\mathbf{f} - 2\lambda\\ \therefore\quad \theta &= \frac{\mathbf{e}^T\mathbf{V}^{-1}\mathbf{f}}{\mathbf{e}^T\mathbf{V}^{-1}\mathbf{e}} - \frac{2\lambda}{\mathbf{e}^T\mathbf{V}^{-1}\mathbf{e}}.\end{aligned}$$

Hence, the holdings of the portfolio are

$$\begin{aligned}\mathbf{h} &= \frac{1}{2\lambda}\mathbf{V}^{-1}\left(\frac{\mathbf{e}^T\mathbf{V}^{-1}\mathbf{f}}{\mathbf{e}^T\mathbf{V}^{-1}\mathbf{e}} - \frac{2\lambda}{\mathbf{e}^T\mathbf{V}^{-1}\mathbf{e}}\mathbf{e}\right)\\ &= \frac{\mathbf{V}^{-1}\mathbf{e}}{\mathbf{e}^T\mathbf{V}^{-1}\mathbf{e}} + \frac{1}{2\lambda}\left(\mathbf{V}^{-1}\mathbf{f} - \frac{\mathbf{e}^T\mathbf{V}^{-1}\mathbf{f}}{\mathbf{e}^T\mathbf{V}^{-1}\mathbf{e}}\mathbf{V}^{-1}\mathbf{e}\right).\end{aligned}$$

From this we can compute the expected excess return of the portfolio, namely

$$\begin{aligned}\mathbf{f}^T\mathbf{h} &= \frac{\mathbf{f}^T\mathbf{V}^{-1}\mathbf{e}}{\mathbf{e}^T\mathbf{V}^{-1}\mathbf{e}} + \frac{1}{2\lambda}\left(\mathbf{f}^T\mathbf{V}^{-1}\mathbf{f} - \frac{(\mathbf{e}^T\mathbf{V}^{-1}\mathbf{f})^2}{\mathbf{e}^T\mathbf{V}^{-1}\mathbf{e}}\right)\\ &= f_C + \frac{1}{2\lambda}\left(\mathbf{f}^T\mathbf{V}^{-1}\mathbf{f} - \frac{(\mathbf{e}^T\mathbf{V}^{-1}\mathbf{f})^2}{\mathbf{e}^T\mathbf{V}^{-1}\mathbf{e}}\right)\\ &= f_C + \frac{1}{2\lambda}\frac{\left(\mathbf{f}^T\mathbf{V}^{-1}\mathbf{f} - \frac{(\mathbf{e}^T\mathbf{V}^{-1}\mathbf{f})^2}{\mathbf{e}^T\mathbf{V}^{-1}\mathbf{e}}\right)\left(\frac{\mathbf{f}^T\mathbf{V}^{-1}\mathbf{f}}{(\mathbf{e}^T\mathbf{V}^{-1}\mathbf{f})^2} - \frac{1}{\mathbf{e}^T\mathbf{V}^{-1}\mathbf{e}}\right)}{\frac{\mathbf{f}^T\mathbf{V}^{-1}\mathbf{f}}{(\mathbf{e}^T\mathbf{V}^{-1}\mathbf{f})^2} - \frac{1}{\mathbf{e}^T\mathbf{V}^{-1}\mathbf{e}}}\\ &= f_C + \frac{1}{2\lambda}\frac{\frac{(\mathbf{f}^T\mathbf{V}^{-1}\mathbf{f})^2}{(\mathbf{e}^T\mathbf{V}^{-1}\mathbf{f})^2} - 2\frac{\mathbf{f}^T\mathbf{V}^{-1}\mathbf{f}}{\mathbf{e}^T\mathbf{V}^{-1}\mathbf{e}} + \frac{(\mathbf{e}^T\mathbf{V}^{-1}\mathbf{f})^2}{(\mathbf{e}^T\mathbf{V}^{-1}\mathbf{e})^2}}{\frac{\mathbf{f}^T\mathbf{V}^{-1}\mathbf{f}}{(\mathbf{e}^T\mathbf{V}^{-1}\mathbf{f})^2} - \frac{1}{\mathbf{e}^T\mathbf{V}^{-1}\mathbf{e}}}\\ &= f_C + \frac{1}{2\lambda}\frac{f_Q^2 - 2f_Qf_C - f_C^2}{\sigma_Q^2 - \sigma_C^2}\\ &= f_C + \frac{1}{2\lambda}\frac{(f_Q - f_C)^2}{\sigma_Q^2 - \sigma_C^2}\\ &= f_C + \frac{1}{2\lambda\kappa},\end{aligned}$$

as desired.

Problem 2a.5. Prove that portfolio Q is the optimal solution in Exercise 4 if $\lambda = \frac{f_C}{2\sigma_C^2} = \frac{f_Q}{2\sigma_Q^2}$.

Solution. By (2A.14) and (2A.15),

$$\frac{f_C}{\sigma_C^2} = \mathbf{f}^T \mathbf{V}^{-1} \mathbf{e}.$$

By the solution to the previous problem, we have

$$\mathbf{h} = \frac{\mathbf{V}^{-1}\mathbf{e}}{\mathbf{e}^T \mathbf{V}^{-1} \mathbf{e}} + \frac{1}{2\lambda}\left(\mathbf{V}^{-1}\mathbf{f} - \frac{\mathbf{e}^T \mathbf{V}^{-1}\mathbf{f}}{\mathbf{e}^T \mathbf{V}^{-1}\mathbf{e}} \mathbf{V}^{-1}\mathbf{e}\right).$$

Hence, if $\lambda = \frac{f_C}{2\sigma_C^2}$, we have that

$$\begin{aligned}\mathbf{h} &= \frac{\mathbf{V}^{-1}\mathbf{e}}{\mathbf{e}^T \mathbf{V}^{-1} \mathbf{e}} + \frac{\mathbf{V}^{-1}\mathbf{f}}{\mathbf{f}^T \mathbf{V}^{-1} \mathbf{e}} - \frac{\mathbf{e}^T \mathbf{V}^{-1}\mathbf{f}}{\mathbf{e}^T \mathbf{V}^{-1}\mathbf{e}} \frac{\mathbf{V}^{-1}\mathbf{e}}{\mathbf{f}^T \mathbf{V}^{-1}\mathbf{e}} \\ &= \frac{\mathbf{V}^{-1}\mathbf{f}}{\mathbf{f}^T \mathbf{V}^{-1}\mathbf{e}},\end{aligned}$$

which are the holdings of portfolio Q, as we noted in the solution to the previous problem.

Problem 2a.6. Suppose portfolio T is on the fully invested efficient frontier. Prove Eq. (2A.45), i.e., that there exists a w_T such that $\mathbf{h}_T = w_T \mathbf{h}_C + (1-w_T)\mathbf{h}_Q$.

Solution. Since T is on the fully invested efficient frontier, T has minimum risk amongst all portfolios with the same expected return f_P. Hence, to find T, we wish to minimise

$$\frac{\mathbf{h}^T \mathbf{V} \mathbf{h}}{2}$$

subject to the constraints

$$\begin{aligned}\mathbf{e}^T \mathbf{h} &= 1 \\ \mathbf{f}^T \mathbf{h} &= f_P.\end{aligned}$$

As always, we use the method of Lagrange multipliers and so consider

$$\frac{\mathbf{h}^T \mathbf{V} \mathbf{h}}{2} + \theta_1(\mathbf{e}^T \mathbf{h} - 1) + \theta_2(\mathbf{f}^T \mathbf{h} - f_P).$$

We hence need to solve the simultaneous equations

$$\begin{aligned}\mathbf{V}\mathbf{h} + \theta_1 \mathbf{e} + \theta_2 \mathbf{f} &= 0, \\ \mathbf{e}^T \mathbf{h} &= 1, \\ \mathbf{f}^T \mathbf{h} &= f_P,\end{aligned}$$

From the first equation we obtain that

$$\mathbf{h} = \mathbf{V}^{-1}(-\theta_1 \mathbf{e} - \theta_2 \mathbf{f}). \tag{1}$$

Substituting this into the other two equations gives

$$\begin{aligned}-\theta_1 \mathbf{e}^T \mathbf{V}^{-1} \mathbf{e} - \theta_2 \mathbf{e}^T \mathbf{V}^{-1} \mathbf{f} &= 1, \\ -\theta_1 \mathbf{f}^T \mathbf{V}^{-1} \mathbf{e} - \theta_2 \mathbf{f}^T \mathbf{V}^{-1} \mathbf{f} &= f_P.\end{aligned}$$

We now have two simultaneous equations for θ_1 and θ_2, which we can solve as follows. We rearrange the first equation for θ_1, obtaining

$$\theta_1 = -\frac{\theta_2 \mathbf{e}^T \mathbf{V}^{-1} \mathbf{f} + 1}{\mathbf{e}^T \mathbf{V}^{-1} \mathbf{e}}.$$

We can then substitute this into the second equation to give

$$\frac{(\theta_2 \mathbf{e}^T\mathbf{V}^{-1}\mathbf{f}+1)\mathbf{e}^T\mathbf{V}^{-1}\mathbf{f}}{\mathbf{e}^T\mathbf{V}^{-1}\mathbf{e}} - \theta_2 \mathbf{f}^T\mathbf{V}^{-1}\mathbf{f} = f_P.$$

We now solve this equation for θ_2.

$$\begin{aligned}
&\theta_2(\mathbf{e}^T\mathbf{V}^{-1}\mathbf{f})^2 + \mathbf{e}^T\mathbf{V}^{-1}\mathbf{f} - \theta_2(\mathbf{f}^T\mathbf{V}^{-1}\mathbf{f})(\mathbf{e}^T\mathbf{V}^{-1}\mathbf{e}) = f_P\mathbf{e}^T\mathbf{V}^{-1}\mathbf{e}\\
\therefore\quad &((\mathbf{e}^T\mathbf{V}^{-1}\mathbf{f})^2 - (\mathbf{f}^T\mathbf{V}^{-1}\mathbf{f})(\mathbf{e}^T\mathbf{V}^{-1}\mathbf{e}))\theta_2 = f_P\mathbf{e}^T\mathbf{V}^{-1}\mathbf{e} - \mathbf{e}^T\mathbf{V}^{-1}\mathbf{f}\\
\therefore\quad &\theta_2 = \frac{f_P\mathbf{e}^T\mathbf{V}^{-1}\mathbf{e} - \mathbf{e}^T\mathbf{V}^{-1}\mathbf{f}}{(\mathbf{e}^T\mathbf{V}^{-1}\mathbf{f})^2 - (\mathbf{f}^T\mathbf{V}^{-1}\mathbf{f})(\mathbf{e}^T\mathbf{V}^{-1}\mathbf{e})}
\end{aligned}$$

This gives θ_1 via our expression for θ_1 in terms of θ_2.

$$\begin{aligned}
\theta_1 &= -\frac{\frac{f_P\mathbf{e}^T\mathbf{V}^{-1}\mathbf{e} - \mathbf{e}^T\mathbf{V}^{-1}\mathbf{f}}{(\mathbf{e}^T\mathbf{V}^{-1}\mathbf{f})^2 - (\mathbf{f}^T\mathbf{V}^{-1}\mathbf{f})(\mathbf{e}^T\mathbf{V}^{-1}\mathbf{e})}\mathbf{e}^T\mathbf{V}^{-1}\mathbf{f}+1}{\mathbf{e}^T\mathbf{V}^{-1}\mathbf{e}}\\
&= -\frac{(f_P\mathbf{e}^T\mathbf{V}^{-1}\mathbf{e} - \mathbf{e}^T\mathbf{V}^{-1}\mathbf{f})\mathbf{e}^T\mathbf{V}^{-1}\mathbf{f} + (\mathbf{e}^T\mathbf{V}^{-1}\mathbf{f})^2 - (\mathbf{f}^T\mathbf{V}^{-1}\mathbf{f})(\mathbf{e}^T\mathbf{V}^{-1}\mathbf{e})}{\mathbf{e}^T\mathbf{V}^{-1}\mathbf{e}((\mathbf{e}^T\mathbf{V}^{-1}\mathbf{f})^2 - (\mathbf{f}^T\mathbf{V}^{-1}\mathbf{f})(\mathbf{e}^T\mathbf{V}^{-1}\mathbf{e}))}\\
&= -\frac{f_P\mathbf{e}^T\mathbf{V}^{-1}\mathbf{f} - \mathbf{f}^T\mathbf{V}^{-1}\mathbf{f}}{(\mathbf{e}^T\mathbf{V}^{-1}\mathbf{f})^2 - (\mathbf{f}^T\mathbf{V}^{-1}\mathbf{f})(\mathbf{e}^T\mathbf{V}^{-1}\mathbf{e})}
\end{aligned}$$

Having solved for θ_1 and θ_2, we now substitute back into (1) to find $\mathbf{h}$.

$$\begin{aligned}
\mathbf{h} &= -\theta_1\mathbf{V}^{-1}\mathbf{e} - \theta_2\mathbf{V}^{-1}\mathbf{f}\\
&= \frac{f_P\mathbf{e}^T\mathbf{V}^{-1}\mathbf{f} - \mathbf{f}^T\mathbf{V}^{-1}\mathbf{f}}{(\mathbf{e}^T\mathbf{V}^{-1}\mathbf{f})^2 - (\mathbf{f}^T\mathbf{V}^{-1}\mathbf{f})(\mathbf{e}^T\mathbf{V}^{-1}\mathbf{e})}\mathbf{V}^{-1}\mathbf{e} + \frac{\mathbf{e}^T\mathbf{V}^{-1}\mathbf{f} - f_P\mathbf{e}^T\mathbf{V}^{-1}\mathbf{e}}{(\mathbf{e}^T\mathbf{V}^{-1}\mathbf{f})^2 - (\mathbf{f}^T\mathbf{V}^{-1}\mathbf{f})(\mathbf{e}^T\mathbf{V}^{-1}\mathbf{e})}\mathbf{V}^{-1}\mathbf{f}\\
&= \frac{f_P(\mathbf{e}^T\mathbf{V}^{-1}\mathbf{f})(\mathbf{e}^T\mathbf{V}^{-1}\mathbf{e}) - (\mathbf{f}^T\mathbf{V}^{-1}\mathbf{f})(\mathbf{e}^T\mathbf{V}^{-1}\mathbf{e})}{(\mathbf{e}^T\mathbf{V}^{-1}\mathbf{f})^2 - (\mathbf{f}^T\mathbf{V}^{-1}\mathbf{f})(\mathbf{e}^T\mathbf{V}^{-1}\mathbf{e})}\frac{\mathbf{V}^{-1}\mathbf{e}}{\mathbf{e}^T\mathbf{V}^{-1}\mathbf{e}} + \frac{(\mathbf{e}^T\mathbf{V}^{-1}\mathbf{f})^2 - f_P(\mathbf{e}^T\mathbf{V}^{-1}\mathbf{e})(\mathbf{e}^T\mathbf{V}^{-1}\mathbf{f})}{(\mathbf{e}^T\mathbf{V}^{-1}\mathbf{f})^2 - (\mathbf{f}^T\mathbf{V}^{-1}\mathbf{f})(\mathbf{e}^T\mathbf{V}^{-1}\mathbf{e})}\frac{\mathbf{V}^{-1}\mathbf{f}}{\mathbf{e}^T\mathbf{V}^{-1}\mathbf{f}}\\
&= \frac{f_P - \frac{\mathbf{f}^T\mathbf{V}^{-1}\mathbf{f}}{\mathbf{e}^T\mathbf{V}^{-1}\mathbf{f}}}{\frac{\mathbf{e}^T\mathbf{V}^{-1}\mathbf{f}}{\mathbf{e}^T\mathbf{V}^{-1}\mathbf{e}} - \frac{\mathbf{f}^T\mathbf{V}^{-1}\mathbf{f}}{\mathbf{e}^T\mathbf{V}^{-1}\mathbf{f}}}\mathbf{h}_C + \frac{\frac{\mathbf{e}^T\mathbf{V}^{-1}\mathbf{f}}{\mathbf{e}^T\mathbf{V}^{-1}\mathbf{e}} - f_P}{\frac{\mathbf{e}^T\mathbf{V}^{-1}\mathbf{f}}{\mathbf{e}^T\mathbf{V}^{-1}\mathbf{e}} - \frac{\mathbf{f}^T\mathbf{V}^{-1}\mathbf{f}}{\mathbf{e}^T\mathbf{V}^{-1}\mathbf{f}}}\mathbf{h}_Q\\
&= \frac{f_P - f_Q}{f_C - f_Q}\mathbf{h}_C + \frac{f_C - f_P}{f_C - f_Q}\mathbf{h}_Q\\
&= \frac{f_Q - f_P}{f_Q - f_C}\mathbf{h}_C + \frac{f_P - f_C}{f_Q - f_C}\mathbf{h}_C.
\end{aligned}$$

We hence recover equation (2A.45).

Problem 2a.7. If T is fully invested and efficient and $T \neq C$, prove that there exists a fully invested efficient portfolio T^* such that $\text{Cov}\{r_T, r_{T^*}\} = 0$.

Solution. By the previous problem, we know that we must be able to write

$$\begin{aligned}
\mathbf{h}_T &= w_T\mathbf{h}_C + (1-w_T)\mathbf{h}_Q\\
\mathbf{h}_{T^*} &= w_{T^*}\mathbf{h}_C + (1-w_{T^*})\mathbf{h}_Q,
\end{aligned}$$

since T and T^* are fully invested efficient portfolios. Hence, we have

$$\begin{aligned}
r_T &= w_T r_C + (1-w_T)r_Q\\
r_{T^*} &= w_{T^*} r_C + (1-w_{T^*})r_Q.
\end{aligned}$$

Our approach is to use these equations to compute $\text{Cov}\{r_T, r_{T^*}\}$ in terms of w_T, w_{T^*}, and constants. Then, by setting this covariance equal to zero, we can solve for w_{T^*} in terms of w_T. This then gives us the desired portfolio T^* which is uncorrelated with T. Thus

$$\begin{aligned}
\text{Cov}\{r_T, r_{T^*}\} &= \text{Cov}\{w_T r_C + (1-w_T) r_Q, w_{T^*} r_C + (1-w_{T^*}) r_Q\} \\
&= w_T w_{T^*}\sigma_C^2 + (1-w_T)(1-w_{T^*})\sigma_Q^2 + ((1-w_T)w_{T^*} + (1-w_{T^*})w_T)\sigma_{C,Q} \\
&= w_T w_{T^*}\sigma_C^2 + (1-w_T)(1-w_{T^*})\frac{f_Q}{f_C}\sigma_C^2 + ((1-w_T)w_{T^*} + (1-w_{T^*})w_T)\sigma_C^2 \qquad \because (2A.17), (2A.35) \\
&= \sigma_C^2\left(w_T w_{T^*} + \frac{f_Q}{f_C} - \frac{f_Q}{f_C}w_T - \frac{f_Q}{f_C}w_{T^*} + \frac{f_Q}{f_C}w_T w_{T^*} + w_{T^*} + w_T - 2w_T w_{T^*}\right) \\
&= \sigma_C^2\left[\left(\frac{f_Q}{f_C} - 1\right)w_T w_{T^*} + \left(1 - \frac{f_Q}{f_C}\right)w_T + \left(1 - \frac{f_Q}{f_C}\right)w_{T^*} + \frac{f_Q}{f_C}\right].
\end{aligned}$$

Since we desire $\text{Cov}\{r_T, r_{T^*}\} = 0$, we must have

$$\begin{aligned}
&\left(\frac{f_Q}{f_C} - 1\right)w_T w_{T^*} + \left(1 - \frac{f_Q}{f_C}\right)w_T + \left(1 - \frac{f_Q}{f_C}\right)w_{T^*} + \frac{f_Q}{f_C} = 0 \\
\iff \quad &\left[\left(\frac{f_Q}{f_C} - 1\right)w_T + \left(1 - \frac{f_Q}{f_C}\right)\right]w_{T^*} = \left(\frac{f_Q}{f_C} - 1\right)w_T - \frac{f_Q}{f_C} \\
\iff \quad &w_{T^*} = \frac{\left(\frac{f_Q}{f_C} - 1\right)w_T - \frac{f_Q}{f_C}}{\left(\frac{f_Q}{f_C} - 1\right)w_T + \left(1 - \frac{f_Q}{f_C}\right)}.
\end{aligned}$$

Thus, if we set w_{T^*} as above, we have $\text{Cov}\{r_T, r_{T^*}\} = 0$. This establishes the existence of such a portfolio T^*.

Note that we must assume $T \neq C$, so that $w_T \neq 1$ and the denominator of the fraction is non-zero.

Problem 2a.8. For any $T \neq C$ on the efficient frontier and any fully invested portfolio P, show that we can write

$$E\{r_P\} = E\{r_{T^*}\} + E\{r_T - r_{T^*}\}\frac{\text{Cov}\{r_P, r_T\}}{\text{Var}\{r_T\}}$$

where T^* is the fully invested efficient portfolio that is uncorrelated with T.

Solution. We proceed as follows. We must assume that $T \neq C$ so that we can apply Problem 2a.7.

$$\begin{aligned}
&E\{r_{T^*}\} + E\{r_T - r_{T^*}\}\frac{\text{Cov}\{r_{P, r_T}\}}{\text{Var}\{r_T\}}\} \\
&\quad = f_{T^*} + (f_T - f_{T^*})\frac{\text{Cov}\{r_P - r_{T^*}, r_T\}}{\text{Cov}\{r_T - r_{T^*}, r_T\}} \quad \because T \text{ and } T^{ast} \text{ are uncorrelated} \\
&\quad = f_{T^*} + (f_T - f_{T^*})\frac{\text{Cov}\{r_P - r_{T^*}, w_T r_C + (1-w_T) r_Q\}}{\text{Cov}\{r_T - r_{T^*}, w_T r_C + (1-w_T) r_Q\}} \\
&\quad = f_{T^*} + (f_T - f_{T^*})\frac{w_T(\text{Cov}\{r_P, r_C\} - \text{Cov}\{r_{T^*}, r_C\}) + (1-w_T)(\text{Cov}\{r_P, r_Q\} - \text{Cov}\{r_{T^*, r_Q}\})}{w_T(\text{Cov}\{r_T, r_C\} - \text{Cov}\{r_{T^*}, r_C\}) + (1-w_T)(\text{Cov}\{r_T, r_Q\} - \text{Cov}\{r_{T^*, r_Q}\})} \\
&\quad = f_{T^*} + (f_T - f_{T^*})\frac{w_T(e_P\sigma_C^2 - e_{T^*}\sigma_C^2) + (1-w_T)(e_q f_P \sigma_Q^2 - e_q f_{T^*}\sigma_Q^2)}{w_T(e_T\sigma_C^2 - e_{T^*}\sigma_C^2) + (1-w_T)(e_q f_T \sigma_Q^2 - e_q f_{T^*}\sigma_Q^2)} \quad \because (2A.4) \\
&\quad = f_{T^*} + (f_T - f_{T^*})\frac{(1-w_T)e_q\sigma_Q^2(f_P - f_{T^*})}{(1-w_T)e_q\sigma_Q^2(f_T - f_{T^*})} \quad \because \text{The portfolios are all fully invested} \\
&\quad = f_{T^*} + (f_P - f_{T^*}) \\
&\quad = f_P.
\end{aligned}$$

Note that $1 - w_T \neq 0$ since $T \neq C$.

Problem 2a.9. If P is any fully invested portfolio and T is the efficient fully invested portfolio with the same expected returns as P, $\mu_P = \mu_T$, we can always write the returns to P as $r_P = r_C + (r_T - r_C) + (r_P - r_T)$. Prove that these three components of return are uncorrelated. We can interpret the risks associated with these three components as the cost of full investment, $\text{Var}\{r_C\}$; the cost of expected return $\mu_P - \mu_C$, $\text{Var}\{r_T - r_C\}$; and the diversifiable cost, $\text{Var}\{r_P - r_T\}$.

Solution. There are three pairs of components that we need to show are uncorrelated and we address each in turn.

$$\begin{aligned} \text{Cov}\{r_C, r_T - r_C\} &= \text{Cov}\{r_C, r_T\} - \text{Cov}\{r_C, r_C\} \\ &= e_T \sigma_C^2 - \sigma_C^2 \qquad (2A.4) \\ &= 0. \end{aligned}$$

Here the last line follows from the fact that T is fully invested. We next consider the following.

$$\begin{aligned} \text{Cov}\{r_C, r_P - r_T\} &= \text{Cov}\{r_C, r_P\} - \text{Cov}\{r_C, r_T\} \\ &= e_P \sigma_C^2 - e_T \sigma_C^2 \qquad (2A.4) \\ &= 0. \end{aligned}$$

Again, the last line follows from the fact that T and P are both fully invested. We now consider the last pair of components.

$$\begin{aligned} \text{Cov}\{r_T - r_C, r_P - r_T\} &= \text{Cov}\{r_T, r_P - r_T\} - \text{Cov}\{r_C, r_P - r_T\} \\ &= \text{Cov}\{r_T, r_P - r_T\}, \end{aligned}$$

by above. By Problem 2.a8, we have

$$\begin{aligned} E\{r_P\} &= E\{r_{T^*}\} + E\{r_T - r_{T^*}\} \frac{\text{Cov}\{r_P, r_T\}}{\text{Var}\{r_T\}} \\ \therefore \quad E\{r_T\}\text{Var}\{r_T\} &= \text{Var}\{r_T\} E\{r_{T^*}\} + E\{r_T - r_{T^*}\}\text{Cov}\{r_P, r_T\} \qquad \because E\{r_T\} = E\{r_P\} \\ \therefore \quad E\{r_T - r_{T^*}\}\text{Var}\{r_T\} &= E\{r_T - r_{T^*}\}\text{Cov}\{r_P, r_T\} \\ \therefore \quad \text{Var}\{r_T\} &= \text{Cov}\{r_P, r_T\} \\ \therefore \quad \text{Cov}\{r_T, r_P - r_T\} &= 0. \end{aligned}$$

This completes the proof.

Chapter 3

Problem 3.1. If GE has an annual risk of 27.4 percent, what is the volatility of monthly GE returns?

Solution. From eq (3.6) we have

$$\sigma_{annual} = \sqrt{12} \times \sigma_{monthly}$$

Hence,

$$\begin{aligned}\sigma^{GE}_{monthly} &= \frac{27.4\%}{\sqrt{12}} \\ &= 7.91\%\end{aligned}$$

Problem 3.2. Stock A has 25 percent risk, stock B has 50 percent risk, and their returns are 50 percent correlated. What fully invested portfolio of A and B has minimum total risk? (*Hint* try solving graphically (e.g. in Excel), if you cannot determine the answer mathematically.)

Solution. The risk of the portfolio will be (see Eq. (3.1))

$$\sigma_P = \sqrt{(f_A\sigma_A)^2 + ((1-f_A)\sigma_B)^2 + 2f_A\sigma_A(1-f_A)\sigma_B\rho_{AB}}$$

where ρ_{AB} (=50%) is the correlation between A and B and f_A is the fraction of the portfolio invested in A. The fully invested constraint, $f_A + f_B = 1$ leads to the $1 - f_A$ term in front of σ_B. To minimize the total risk, we solve

$$\frac{\partial \sigma_P}{\partial f_A} = 0$$

for f_A. We have

$$\frac{\partial \sigma_P}{\partial f_A} = \frac{1}{2}\frac{2f_A\sigma_A^2 - 2(1-f_A)\sigma_B^2 + (2-4f_A)\sigma_A\sigma_B\rho_{AB}}{\sqrt{(f_A\sigma_A)^2 + ((1-f_A)\sigma_B)^2 + 2f_A\sigma_A(1-f_A)\sigma_B\rho_{AB}}} \tag{2}$$

Setting the numerator to zero, we have

$$\begin{aligned}0 &= 2f_A\sigma_A^2 - 2(1-f_A)\sigma_B^2 + (2-4f_A)\sigma_A\sigma_B\rho_{AB} \\ &= 2f_A(\sigma_A^2+\sigma_B^2) - 4f_A\sigma_A\sigma_B\rho_{AB} - 2\sigma_B^2 + 2\sigma_A\sigma_B\rho_{AB} \\ 2\sigma_B^2 - 2\sigma_A\sigma_B\rho_{AB} &= f_A(2\sigma_A^2 + 2\sigma_B^2 - 4\sigma_A\sigma_B\rho_{AB}) \\ f_A &= \frac{2\sigma_B^2 - 2\sigma_A\sigma_B\rho_{AB}}{2\sigma_A^2 + 2\sigma_B^2 - 4\sigma_A\sigma_B\rho_{AB}}\end{aligned}$$

Plugging in the values, we have

$$\begin{aligned}f_A &= \frac{2(0.25) - 2(0.5)(0.25)(0.5)}{2(0.0625) + 2(0.25) - 4(0.5)(0.25)(0.5)} \\ &= \frac{0.375}{0.375} \\ &= 1\end{aligned}$$

Hence, the portfolio with minimum risk will hold 100% stock A.

Solution (NJW). One can also solve this problem using Lagrange multipliers. The covariance matrix for stock A and stock B is

$$\begin{aligned}\begin{pmatrix}\sigma_A^2 & \text{Cov}\{r_A, r_B\} \\ \text{Cov}\{r_A, r_B\} & \sigma_A^2\end{pmatrix} &= \begin{pmatrix}\sigma_A^2 & \rho_{AB}\sigma_A\sigma_B \\ \rho_{AB}\sigma_A\sigma_B & \sigma_A^2\end{pmatrix} \\ &= \begin{pmatrix}25^2 & 0.5\times 25\times 50 \\ 0.5\times 25\times 50 & 50^2\end{pmatrix} \\ &= \begin{pmatrix}625 & 625 \\ 625 & 2500\end{pmatrix}.\end{aligned}$$

Note that we follow the convention of Grinold and Kahn convention of representing risk and variance in decimal, rather than as percentages. (See the brief discussion and calculations at (4.12) on p.97.)

Minimising risk is the same as minimising variance, so we simplify by doing the latter. Let a and b be the respective holdings in stocks A and B. Our Lagrangian function is then

$$\begin{aligned}\begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} 625 & 625 \\ 625 & 2500 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} - \theta(a+b-1) \\ = 625a^2 + 1250ab + 2500b^2 - \theta(a+b-1).\end{aligned}$$

By setting the three partial derivatives of this function equal to zero, we obtain the following equations.

$$\begin{aligned}1250a + 1250b - \theta &= 0 \\ 1250a + 5000b - \theta &= 0 \\ a + b &= 1.\end{aligned}$$

It is clear from the first two equations that $b = 0$, and so $a = 1$. Hence the fully invested portfolio of A and B with minimum total risk has 100% of its holdings in A and 0% of its holdings in B.

Problem 3.3. What is the risk of an equal-weighted portfolio consisting of five stocks, each with 35 percent volatility and a 50 percent correlation with all other stocks? How does that increase as the portfolio increases to 20 stocks or 100 stocks?

Solution. From eq. (3.4) we have

$$\sigma_P = \sigma\sqrt{\frac{1+\rho(N-1)}{N}}$$

Hence, for 5, 20, 100 and an infinite number of stocks, we have

$$\begin{aligned}\sigma_P^{N=5} &= 27.1\% \\ \sigma_P^{N=20} &= 25.4\% \\ \sigma_P^{N=100} &= 24.9\% \\ \sigma_P^{N=100} &= 24.7\%\end{aligned}$$

Problem 3.4. How do structural risk models help in estimating asset betas? How do these betas differ from those estimated from a 60-month beta regression?

Solution. The beta of asset M is defined relative to the benchmark as

$$\beta_M = \frac{\sigma_{M,B}}{\sigma_B^2}$$

Structural risk models allow us to predict the risk of and correlations between stocks from which it is straightforward to calculate asset betas. In this chapter, the authors highlight the pros of using structural risk models and the cons of using regressions from historical data. The pros of structural risk models are

- The size of the problem can be greatly reduced. Instead of dealing with individual stocks and correlations between them, we deal only with factors and correlations between the factors. The stocks can then be projected onto the lower dimensional space of the factors.
- The use of factors allows the actual stocks to change. We only need the exposures of the stocks to the factors

The cons of historical regressions are:

- Dividends, splits, and mergers are hard to account for
- There is selection bias as failed companies are omitted
- In general, the number of observations must be greater than the number of stocks. Hence, for a 60 month beta regression, the observations would have to be daily or weekly, while the forecast would likely be quarterly or yearly.
- These models will take much longer to analyze since there are many more stocks than there are risk factors for the factor models

Chapter 3 technical appendix

Problem 3a.1. Show that:

$$\mathbf{h}_P^T \cdot \mathbf{MCTR} = \sigma_P$$
$$\mathbf{h}_P^T \cdot \mathbf{MCRR} = \omega_P$$
$$\mathbf{h}_{PA}^T \cdot \mathbf{MCAR} = \psi_P$$

Solution. We show each in turn.

$$\begin{aligned}
\mathbf{h}_P^T \cdot \mathbf{MCTR} &= \frac{\mathbf{h}_P^T \mathbf{V} \mathbf{h}_P}{\sigma_P} && (3A.20)\\
&= \frac{\sigma_P^2}{\sigma}\\
&= \sigma_P.\\
\mathbf{h}_P^T \cdot \mathbf{MCRR} &= \frac{\mathbf{h}_P^T \mathbf{V} \mathbf{h}_{PR}}{\omega_P} && (3A.22)\\
&= \frac{\mathbf{h}_P^T \mathbf{V} \mathbf{h}_P - \mathbf{h}_P^T \mathbf{V} \mathbf{h}_B}{\omega_P}\\
&= \frac{\sigma_P^2 - \beta_P \sigma_B^2}{\omega_P} && (2.1)\\
&= \frac{\omega_P^2}{\omega_P} && (3.8)\\
&= \omega_P.\\
\mathbf{h}_{PA}^T \cdot \mathbf{MCAR} &= \frac{\mathbf{h}_{PA}^T \mathbf{V} \mathbf{h}_{PA}}{\psi_P} && (3A.23)\\
&= \frac{\psi_P^2}{\psi_P} && (3A.12)\\
&= \psi_P.
\end{aligned}$$

Note that most of the steps of the final derivation are carried out at (3A.31).

Problem 3a.2. Verify Eq. (3A.24)

Solution. Eq. (3A.24) states that

$$\mathbf{MCAR} = \boldsymbol{\beta} k_1 + \mathbf{MCRR} k_2$$

where

$$k_1 = \frac{\beta_{PA} \sigma_B^2}{\psi_P}$$

and

$$k_2 = \frac{\omega_P}{\psi_P}.$$

We can derive this as follows.

$$
\begin{aligned}
\boldsymbol{\beta}\frac{\beta_{PA}\sigma_B^2}{\psi_P} + \frac{\mathbf{V}\mathbf{h}_{PR}}{\omega_P}\frac{\omega_P}{\psi_P} &= \frac{1}{\psi_P}\left(\frac{\mathbf{V}\mathbf{h}_B}{\sigma_B^2}\beta_{PA}\sigma_B^2 + \mathbf{V}\mathbf{h}_{PR}\right) \\
&= \frac{1}{\psi_P}(\beta_{PA}\mathbf{V}\mathbf{h}_B + \mathbf{V}\mathbf{h}_{PR}) \\
&= \frac{1}{\psi_P}(\beta_{PA}\mathbf{V}\mathbf{h}_B + \mathbf{V}(\mathbf{h}_P - \beta_P\mathbf{h}_B)) \\
&= \frac{1}{\psi_P}((\beta_{PA} - \beta_P)\mathbf{V}\mathbf{h}_B + \mathbf{V}\mathbf{h}_P) \\
&= \frac{1}{\psi_P}(-\mathbf{V}\mathbf{h}_B + \mathbf{V}\mathbf{h}_P) \\
&= \frac{\mathbf{V}(\mathbf{h}_P - \mathbf{h}_B)}{\psi_P} \\
&= \frac{\mathbf{V}\mathbf{h}_{PA}}{\psi_P} \\
&= \mathbf{MCAR}.
\end{aligned}
$$

Problem 3a.3. Show that

$$
\begin{aligned}
\mathbf{h}_B^T \cdot \mathbf{MCRR} &= 0 \\
\mathbf{h}_B^T \cdot \mathbf{MCAR} &= k_1.
\end{aligned}
$$

Solution. We derive these two identities in turn.

$$
\begin{aligned}
\mathbf{h}_B^T \cdot \mathbf{MCRR} &= \frac{\mathbf{h}_B^T\mathbf{V}\mathbf{h}_{PR}}{\omega_P} \\
&= \frac{\mathbf{h}_B^T\mathbf{V}(\mathbf{h}_P - \beta_P\mathbf{h}_P)}{\omega_P} \\
&= \frac{\mathbf{h}_B^T\mathbf{V}\mathbf{h}_P^T - \beta_P\mathbf{h}_B^T\mathbf{V}\mathbf{h}_B}{\omega_P} \\
&= \frac{\mathbf{h}_B^T\mathbf{V}\mathbf{h}_P - \mathbf{h}_B^T\mathbf{V}\mathbf{h}_P}{\omega_P} \\
&= 0. \\
\mathbf{h}_P^T \cdot \mathbf{MCAR} &= \frac{\mathbf{h}_B^T\mathbf{V}\mathbf{h}_{PA}}{\psi_P} \\
&= \frac{\frac{\mathbf{h}_B^T\mathbf{V}\mathbf{h}_{PA}}{\sigma_B^2}\sigma_B^2}{\psi_P} \\
&= \frac{\beta_{PA}\sigma_B^2}{\psi_P} \\
&= k_1.
\end{aligned}
$$

Problem 3a.4. Using the single-factor model, assuming that every stock has equal residual risk ω_0, and considering equal-weighted portfolios to track the equal-weighted S&P 500, show that the residual risk of the N-stock portfolio will be

$$\omega_N^2 = \frac{\omega_0^2}{N}.$$

What estimate does this provide of how well a 50-stock portfolio could track the S&P 500? Assume $\omega_0 =$ 25 percent.

Solution. Let r_N be the random variable of the excess return of the N-stock portfolio, so that $r_N = \frac{1}{N}\sum_{i=1}^{N} r_{iN}$, where r_{iN} are the random variables for the excess returns of the individual stocks in the portfolio. We then have

$$\begin{aligned}\sigma_N^2 &:= \operatorname{Var}\{r_N\} \\ &= \operatorname{Cov}\left\{\frac{1}{N}\sum_{i=1}^{N} r_{iN}, \frac{1}{N}\sum_{i=1}^{N} r_{iN}\right\} \\ &= \frac{1}{N^2}\sum_{i=1}^{N}\operatorname{Var}\{r_{iN}\} + \frac{1}{N^2}\sum_{i\neq j}\operatorname{Cov}\{r_{iN}, r_{jN}\}.\end{aligned}$$

We let σ_B^2 be the variance of the S&P 500. By the single factor model ((3.12) and (3.13)), and since we assume that every stock has equal residual risk ω_0, we have that

$$\begin{aligned}\sigma_{iN}^2 := \operatorname{Var}\{r_{iN}\} &= \beta_{iN}\sigma_B^2 + \omega_0^2 \\ \operatorname{Cov}\{r_{iN}, r_{jN}\} &= \beta_{iN}\beta_{jN}\sigma_B^2,\end{aligned}$$

where β_{iN}, β_{jN} are the respective betas of the stocks with respect to the S&P 500. Since we assume equal-weighted portfolios to track the equal weighted S&P 500, we can assume that $\beta_{iN} = 1$ for all i. We then continue to reason

$$\begin{aligned}\sigma_N^2 &= \frac{1}{N^2}\sum_{i=1}^{N}\sigma_0^2 + \frac{1}{N^2}\sum_{i\neq j}\sigma_B^2 \\ &= \frac{1}{N^2}N\sigma_0^2 + \frac{1}{N^2}N(N-1)\sigma_B^2 \\ &= \frac{1}{N}(\sigma_B^2 + \omega_0^2) + \frac{N-1}{N}\sigma_B^2 \\ &= \sigma_B^2 + \frac{\omega_0^2}{N}.\end{aligned}$$

It is then clear from the single-factor model applied to the N-stock portfolio that $\omega_N^2 = \frac{\omega_0^2}{N}$.

The measure of how well an N-stock portfolio could track the S&P 500 is the tracking error ψ_N, where

$$\begin{aligned}\psi_N^2 &= \operatorname{Var}\{r_N - r_B\} \\ &= \sigma_N^2 + \sigma_B^2 - 2\operatorname{Cov}\{r_N, r_B\} \\ &= \sigma_N^2 + \sigma_B^2 - 2\beta_N\sigma_B^2.\end{aligned}$$

Since we are assuming $\beta_N = 1$, we conclude that $\psi_N^2 = \sigma_N^2 - \sigma_B^2 = \omega_N^2$. Hence, by the previous part of the question, we deduce that

$$\begin{aligned}\psi_{50} &= \sqrt{\frac{\omega_0^2}{50}}\% \\ &= \frac{25}{\sqrt{50}}\% \\ &= 3.54\%.\end{aligned}$$

This is our estimate of how well a 50-stock portfolio could track the S&P 500.

Problem 3a.5. This is for prime-time players. Show that the inverse of $\mathbf{V}$ is given by

$$\mathbf{V}^{-1} = \boldsymbol{\Delta}^{-1} - \boldsymbol{\Delta}^{-1}\cdot X\cdot(\mathbf{X}^T\cdot\boldsymbol{\Delta}^{-1}\cdot\mathbf{X} + \mathbf{F}^{-1})^{-1}\cdot\mathbf{X}^T\cdot\boldsymbol{\Delta}^{-1}.$$

As we will see in later chapters, portfolio construction problems typically involve inverting the covariance matrix. This useful relationship facilitates that computation by replacing the inversion of an N by N matrix with the inversion of K by K matrices, where $K \ll N$. Note that the inversion of N by N diagonal matrices is trivial.

Solution. We first use (3A.2), which says that $\mathbf{V} = \mathbf{X} \cdot \mathbf{F} \cdot \mathbf{X}^T + \mathbf{\Delta}$. We denote the N by N identity matrix by $\mathbf{I}$. Hence,

$$
\begin{aligned}
&\mathbf{V}\left(\mathbf{\Delta}^{-1} - \mathbf{\Delta}^{-1} \cdot \mathbf{X} \cdot (\mathbf{X}^T \cdot \mathbf{\Delta}^{-1} \cdot \mathbf{X} + \mathbf{F}^{-1})^{-1} \cdot \mathbf{X}^T \cdot \mathbf{\Delta}^{-1}\right) \\
&\quad = \left(\mathbf{X} \cdot \mathbf{F} \cdot \mathbf{X}^T + \mathbf{\Delta}\right)\left(\mathbf{\Delta}^{-1} - \mathbf{\Delta}^{-1} \cdot \mathbf{X} \cdot (\mathbf{X}^T \cdot \mathbf{\Delta}^{-1} \cdot \mathbf{X} + \mathbf{F}^{-1})^{-1} \cdot \mathbf{X}^T \cdot \mathbf{\Delta}^{-1}\right) \\
&\quad = \mathbf{X} \cdot \mathbf{F} \cdot \mathbf{X}^T \cdot \mathbf{\Delta}^{-1} - \mathbf{X} \cdot \mathbf{F} \cdot \mathbf{X}^T \cdot \mathbf{\Delta}^{-1} \cdot \mathbf{X} \cdot \left(\mathbf{X}^T \cdot \mathbf{\Delta}^{-1} \cdot \mathbf{X} + \mathbf{F}^{-1}\right)^{-1} \cdot \mathbf{X}^T \cdot \mathbf{\Delta}^{-1} \\
&\qquad + \mathbf{I} - \mathbf{X} \cdot \left(\mathbf{X}^T \cdot \mathbf{\Delta}^{-1} \cdot \mathbf{X} + \mathbf{F}^{-1}\right)^{-1} \cdot \mathbf{X}^T \cdot \mathbf{\Delta}^{-1} \\
&\quad = \mathbf{I} + \mathbf{X} \cdot \left[\mathbf{F} - \mathbf{F} \cdot \mathbf{X}^T \cdot \mathbf{\Delta}^{-1} \cdot \mathbf{X} \cdot \left(\mathbf{X}^T \cdot \mathbf{\Delta}^{-1} \cdot \mathbf{X} + \mathbf{F}^{-1}\right)^{-1} - \left(\mathbf{X}^T \cdot \mathbf{\Delta}^{-1} \cdot \mathbf{X} + \mathbf{F}^{-1}\right)^{-1}\right] \cdot \mathbf{X}^T \cdot \mathbf{\Delta}^{-1} \\
&\quad = \mathbf{I} + \mathbf{X} \cdot \mathbf{F} \cdot \left[\mathbf{I} - \mathbf{X}^T \cdot \mathbf{\Delta}^{-1} \cdot \mathbf{X} \cdot \left(\mathbf{X}^T \cdot \mathbf{\Delta}^{-1} \cdot \mathbf{X} + \mathbf{F}^{-1}\right)^{-1} - \mathbf{F}^{-1}\left(\mathbf{X}^T \cdot \mathbf{\Delta}^{-1} \cdot \mathbf{X} + \mathbf{F}^{-1}\right)^{-1}\right] \cdot \mathbf{X}^T \cdot \mathbf{\Delta}^{-1} \\
&\quad = \mathbf{I} + \mathbf{X} \cdot \mathbf{F} \cdot \left[\mathbf{I} - \left(\mathbf{X}^T \cdot \mathbf{\Delta}^{-1} \cdot \mathbf{X} + \mathbf{F}^{-1}\right)\left(\mathbf{X}^T \cdot \mathbf{\Delta}^{-1} \cdot \mathbf{X} + \mathbf{F}^{-1}\right)^{-1}\right] \cdot \mathbf{X}^T \cdot \mathbf{\Delta}^{-1} \\
&\quad = \mathbf{I} - \mathbf{X} \cdot \mathbf{F} \cdot (\mathbf{I} - \mathbf{I})\mathbf{X}^T \mathbf{\Delta}^{-1} \\
&\quad = \mathbf{I}.
\end{aligned}
$$

Showing that

$$\left(\mathbf{\Delta}^{-1} - \mathbf{\Delta}^{-1} \cdot \mathbf{X} \cdot (\mathbf{X}^T \cdot \mathbf{\Delta}^{-1} \cdot \mathbf{X} + \mathbf{F}^{-1})^{-1} \cdot \mathbf{X}^T \cdot \mathbf{\Delta}^{-1}\right)\mathbf{V} = \mathbf{I}$$

is symmetrical.

Chapter 4

Problem 4.1. Assume a risk-free rate of 6 percent, a benchmark expected excess return of 6.5 percent, and a long range benchmark expected excess return of 6 percent. Given that McDonald's has a beta of 1.07 and an expected total return of 15 percent, separate its expected return into (a) time premium (b) risk premium (c) exceptional benchmark return (d) alpha (e) consensus expected return (f) expected excess return (g) exceptional expected return. (h) What is the sum of the consensus expected return and the exceptional expected return?

Solution. From eq (4.7) we have the total expected return for stock n broken into components as

$$E\{R_n\} = 1 + i_F + \beta_n \mu_B + \beta_n \Delta f_B + \alpha_n$$

For McDonald's stock:

(a) The time premium is just the risk free rate, $i_F = 6\%$.

(b) The risk premium is $\beta_{McDonald's}\mu_B = 1.07 \cdot 6\% = 6.42\%$ where μ_B is the long range expected excess return of the benchmark.

(c) The exceptional benchmark return is $\beta_{McDonald's}\Delta f_B = 1.07 \cdot (6.5 - 6)\% = 0.535\%$ where Δf_B is the difference between the (immediate) expected excess return of the benchmark and the long run expected excess return of the benchmark

(d) Alpha can be found by solving the above equation and plugging in all of the values. We have

$$\begin{aligned}\alpha_{McDonald's} &= E\{R_{McDonald's}\} - 1 - i_F - \beta_n \mu_B - \beta_n \Delta f_B \\ &= 1.15 - 1 - 0.06 - 0.0642 - 0.00535 \\ &= 0.02045\end{aligned}$$

(e) The consensus expected return is just $\beta_{McDonald's} \cdot \mu_B = 6.42\%$

(f) The expected excess return is $f_{McDonald's} = E\{R_{McDonald's}\} - 1 - i_F = 1.15 - 1 - 0.06 = 0.09$ or 9 percent

(g) The exceptional expected return is $f_{McDonald's} - \beta_{McDonald's}\mu_B = 0.09 - 0.0642 = 0.0258$ or 2.58 percent.

(h) The sum of the consensus expected return and the exceptional return is $6.42\% + 2.58\% = 9\%$ which is the expected excess return.

Problem 4.2. Suppose the benchmark is not the market, and the CAPM holds. How will the CAPM expected returns split into the categories suggested in this chapter?

Solution. The CAPM expected excess returns of stock n are equal to $\beta_n^M \mu_M$ where μ_M is the expected excess return of the market. Using eq (4.7), we can set

$$E\{R_n\} = 1 + i_F + \beta_n^M \mu_M = 1 + i_F + \beta_n^B \mu_B + \beta_n^B \Delta f_B + \alpha_n.$$

Here the superscripts indicate the market (M) and benchmark (B). For example β_n^B is the beta of stock n with respect to the benchmark while β_n^M is the beta with respect to the market.

If we suppose that the CAPM holds and that the benchmark is not the market, then we have $\mu_B = \beta_B^M \mu_M$. We also have that $\Delta f_B = 0$, since the near future expected benchmark return f_B is also equal to $\beta_B^M \mu_M$. Hence

$$E\{R_n\} = 1 + i_F + \beta_n^M \mu_M = 1 + i_F + \beta_n^B \beta_B^M \mu_M + \alpha_n.$$

Therefore, the CAPM expected returns will split as follows into the categories suggested in this chapter.

- The time premium will still be equal to i_F.
- The risk premium will equal $\beta_n^B \beta_B^M \mu_M$, which is also equal to the consensus expected return.
- The exceptional benchmark return will be zero.
- Alpha will be $(\beta_B^M - \beta_n^B \beta_B^M)\mu_M$.
- The expected excess return will be $\beta_n^M \mu_M$.
- The exceptional expected return will be equal to the alpha, $(\beta_B^M - \beta_n^B \beta_B^M)\mu_M$.

Problem 4.3. Given a benchmark risk of 20 percent and a portfolio risk of 21 percent, and assuming a portfolio beta of 1, what is the portfolio's residual risk? What is its active risk? How does this compare to the difference between the portfolio risk and the benchmark risk?

Solution. From Eq. (3.13), the variance of a stock n is given by

$$\sigma_n^2 = \beta_n^2 \sigma_B^2 + \omega_n^2$$

so the variance of the portfolio $\boldsymbol{h_P}$ is given by

$$\begin{aligned} \boldsymbol{h_P^T} \cdot \boldsymbol{\sigma^2} &= \boldsymbol{h_P^T} \cdot \boldsymbol{\beta^2} \sigma_B^2 + \boldsymbol{h_P^T} \cdot \boldsymbol{\omega^2} \\ \sigma_P^2 &= \beta_P^2 \sigma_B^2 + \omega_P^2 \end{aligned}$$

where $\boldsymbol{\beta^2}$ is the vector of stock betas squared and $\boldsymbol{\omega^2}$ is the vector of stock residual returns squared. To find the portfolio's residual risk, we can solve for ω_P as.

$$\begin{aligned} \omega_P &= \sqrt{\sigma_P^2 - \beta_P^2 \sigma_B^2} \\ &= \sqrt{(21\%)^2 - 1^2 \times (20\%)^2} \\ &= 6.40\% \end{aligned}$$

The active risk is given by eq (4.20) as

$$\psi_P = \sqrt{\omega_P^2 + \beta_{PA}^2 \cdot \sigma_B^2}$$

The active beta is given by $\beta_{PA} = \beta_P - \beta_B = 1 - 1 = 0$. Hence, the active risk is equal to the residual risk at 6.40%. The active risk is 6.4 percent compared to the difference of risk between the portfolio and the benchmark of 1 percent. The active risk is much larger than the simple difference in portfolio and benchmark risks.

Problem 4.4. Investor A manages total return and risk $(f_P - \lambda_T \cdot \sigma_P^2)$ with risk aversion $\lambda_T = 0.0075$. Investor B manages residual risk and return $(\alpha_P - \lambda_R \cdot \omega_P^2)$, with risk aversion $\lambda_R = 0.075$ (moderate to aggressive). They each can choose between two portfolios:

$$\begin{aligned} f_1 &= 10\% \\ \sigma_1 &= 20.22\% \\ f_2 &= 16\% \\ \sigma_2 &= 25\% \end{aligned}$$

Both portfolios have $\beta = 1$. Furthermore,

$$\begin{aligned} f_B &= 6\% \\ \sigma_B &= 20\% \end{aligned}$$

Which portfolio will A prefer? Which portfolio will B prefer? (*Hint:* First calculate expected residual return and residual risk for the two portfolios.)

Solution. The residual returns are

$$\begin{aligned}\alpha_1 &= f_1 - \beta_1 f_B \\ &= 10\% - 1 \times 6\% \\ &= 4\% \\ \alpha_2 &= f_2 - \beta_2 f_B \\ &= 16\% - 1 \times 6\% \\ &= 10\%\end{aligned}$$

The residual risks are

$$\begin{aligned}\omega_1 &= \sqrt{\sigma_1^2 - \beta_1 \sigma_B^2} \\ &= \sqrt{20.22\%^2 - 1 \times 20\%^2} \\ &= 2.975\% \\ \omega_2 &= \sqrt{\sigma_2^2 - \beta_2 \sigma_B^2} \\ &= \sqrt{25\%^2 - 1 \times 20\%^2} \\ &= 15\%\end{aligned}$$

Investor A will prefer the portfolio with maximum $f_P - \lambda_T \cdot \sigma_P^2$ (highest utility). We have

$$\begin{aligned}f_1 - \lambda_T \cdot \sigma_1^2 &= 10\% - 0.0075 \times 20.22^2\% \\ &= 10\% - 3.07\% \\ &= 6.93\% \\ f_2 - \lambda_T \cdot \sigma_2^2 &= 16\% - 0.0075 \times 25^2\% \\ &= 16\% - 4.69\% \\ &= 11.31\%\end{aligned}$$

so that investor A will prefer portfolio 2. On the other hand, investor B while refer the portfolio with maximum $\alpha_P - \lambda_R \cdot \omega_P^2$. We have

$$\begin{aligned}\alpha_1 - \lambda_R \cdot \omega_1^2 &= 4\% - 0.075 \times 2.975^2\% \\ &= 4\% - 0.65\% \\ &= 3.35\% \\ \alpha_2 - \lambda_R \cdot \omega_2^2 &= 10\% - 0.075 \times 15^2\% \\ &= 10\% - 16.88\% \\ &= -6.88\%\end{aligned}$$

so that investor B will prefer portfolio 1.

Problem 4.5. Assume that you are a mean/variance investor with total risk aversion of 0.0075. If a portfolio has an expected excess return of 6 percent and risk of 20 percent, what is your *certainty equivalent return*, the certain expected excess return that you would fairly trade for this portfolio.

Solution. The certainty equivalent return would be equal to the utility (see p 121) $f_P - \lambda_T \cdot \sigma_P^2$. We have

$$\begin{aligned}f_P - \lambda_T \cdot \sigma_P^2 &= 6\% - .0075 \cdot 20^2\% \\ &= 6\% - 3\% \\ &= 3\%\end{aligned}$$

Chapter 4 technical appendix

Problem 4a.1. Derive the benchmark timing result:

$$\beta_{PA} = \frac{\Delta f_B}{\mu_B}$$

Solution. The question is a little terse. What it wants us to derive is equation (4.14), which says that for the portfolio P which has the highest risk-adjusted return, we have

$$\beta_P = 1 + \frac{\Delta f_B}{\mu_B}.$$

It can be seen that this is equivalent to the equation in the question.

The portfolio P is the portfolio which maximises the expected utility

$$U[P] = f_P - \lambda_T \cdot \sigma_P^2. \tag{4.11}$$

As per the discussion on p.98, the portfolio P will be a mixture of Q and F. Hence let the holdings of portfolio P in risky assets be $\gamma \mathbf{h}_Q$. Here γ is therefore simply the fraction of portfolio P invested in portfolio Q, with the remainder as cash.

We wish to optimise $U[P]$ with respect to γ and so we expand $U[P]$ in those terms:

$$\begin{aligned} U[P] &= f_P - \lambda_T \cdot \sigma_P^2 \\ &= \gamma \cdot f_Q - \lambda_T \cdot \gamma^2 \cdot \sigma_Q^2. \end{aligned}$$

We set the derivative $\frac{dU}{d\gamma}$ equal to zero, obtaining

$$\begin{aligned} f_Q - 2\lambda_T \cdot \gamma \cdot \sigma_Q^2 &= 0 \\ \therefore \quad \gamma &= \frac{f_Q}{2 \cdot \lambda_T \cdot \sigma_Q^2}. \end{aligned}$$

The holdings of portfolio P are therefore

$$\frac{f_Q}{2 \cdot \lambda_T \cdot \sigma_Q^2} \mathbf{h}_Q,$$

as stated in the footnote on p.98.

We can now compute β_P:

$$\begin{aligned} \beta_P &= \frac{f_Q}{2 \cdot \lambda_T \cdot \sigma_Q^2} \beta_Q \\ &= \frac{f_Q}{2 \cdot \lambda_T \cdot \sigma_Q^2} \frac{f_B \sigma_Q^2}{f_Q \sigma_B^2} && \text{by (2A.37)} \\ &= \frac{f_B}{2 \cdot \lambda \cdot \sigma_B^2}. \end{aligned}$$

This is equation (4.13). We then apply equation (4.12), which states that

$$\lambda_T = \frac{\mu_B}{2 \cdot \sigma_B^2}.$$

This gives

$$\beta_P = \frac{f_B}{2 \cdot \lambda \cdot \sigma_B^2} = \frac{f_B}{\mu_B} = 1 + \frac{\Delta f_B}{\mu_B}.$$

Hence

$$\beta_{PA} = \frac{\Delta f_B}{\mu_B},$$

as desired.

Chapter 5

Problem 5.1. What is the information ratio of a passive manager?

Solution. Passive managers will just invest in the benchmark and so their residual returns and risk will be zero. The information ratio will therefore be zero by definition.

Problem 5.2. What is the information ratio required to add a risk-adjusted return of 2.5 precent with a moderate risk aversion level of 0.10? What level of active risk would that require?

Solution. We want to find the IR consistent with a rigk-adjusted value added of 2.5% and a risk aversion of 0.10. From eq. (5.12) we have

$$\text{VA}^* = \frac{\text{IR}^2}{4\lambda_R}$$

Which implies

$$\begin{aligned} \text{IR} &= 2\sqrt{\lambda_R \text{VA}^2} \\ &= 2\sqrt{0.1 \times 2.5\%} \\ &= 1 \end{aligned}$$

From eq. (5.10) we can then calculate the active risk as

$$\begin{aligned} \omega^* &= \frac{\text{IR}}{2\lambda_R} \\ &= \frac{1}{2 * .1} \\ &= 5\% \end{aligned}$$

Problem 5.3. Starting with the universe of MMI stocks, we make the assumptions

$Q =$ MMI portfolio

$f_q = 6\%$

$B =$ capitalization-weighted MMI portfolio

We calculate (as of January 1995) that

Portfolio	$\boldsymbol{\beta}$ **with Respect to** B	$\boldsymbol{\beta}$ **with Respect to** Q	$\boldsymbol{\sigma}$
B	1.000	0.965	15.50%
Q	1.004	1.000	15.82%
C	0.865	0.831	14.42%

where portfolio C is the minimum-variance (fully invested) portfolio. For each portfolio (Q, B, and C), calculate f, α, ω, SR, and IR.

Solution. For portfolio B we have:

$$\begin{aligned} \alpha_B &= \omega_B = 0 \quad \text{(since B is the benchmark)} \\ f_B &= \beta_{B,Q} f_Q + \alpha_B \\ &= 0.965 \times 6\% - 0 \\ &= 5.79\% \\ \text{SR} &= f_B/\sigma_B \\ &= 5.79\%/15.50\% \\ &= 0.374 \\ \text{IR} &= \alpha_B/\omega_B \\ &= 0 \quad \text{(by definition)} \end{aligned}$$

For portfolio Q we have:

$$
\begin{aligned}
f_Q &= 6\% \quad \text{(by definition)} \\
\alpha_Q &= f_Q - \beta_{Q,B} f_B \\
&= 6\% - 1.004 \times 5.79\% \\
&= 0.187\% \\
\omega_Q &= \sqrt{\sigma_Q^2 - \beta_{Q,B}^2 \sigma_B^2} \\
&= \sqrt{(15.82\%)^2 - 1.004^2 \times (15.5\%)^2} \\
&= 2.85\% \\
\text{SR} &= f_Q / \sigma_Q \\
&= 6\%/15.82\% \\
&= 0.379 \\
\text{IR} &= \alpha_Q / \omega_Q \\
&= 0.187\%/2.85\% \\
&= 0.066
\end{aligned}
$$

For portfolio C we have:

$$
\begin{aligned}
f_C &= \beta_{C,B} f_B / \beta_{Q,B} \quad \text{(see Eq 2A.38)} \\
&= 0.865 \times 5.79\%/1.004 \\
&= 4.99\% \\
\alpha_C &= f_C - \beta_{C,B} f_B \\
&= 4.99\% - 0.865 \times 5.79\% \\
&= -0.018\% \\
\omega_C &= \sqrt{\sigma_C^2 - \beta_{C,B}^2 \sigma_B^2} \\
&= \sqrt{(14.42\%)^2 - 0.865^2 \times (15.50\%)^2} \\
&= 5.31\% \\
\text{SR} &= f_C / \sigma_C \\
&= 4.99\%/14.42\% \\
&= 0.346 \\
\text{IR} &= \alpha_C / \omega_C \\
&= -0.018\%/5.31\% \\
&= -0.0034\%
\end{aligned}
$$

Problem 5.4. You have a residual risk aversion of $\lambda_R = 0.12$ and an information ratio of $\text{IR} = 0.60$. What is your optimal level of residual risk? What is your optimal value added?

Solution. From eq. (5.10), the optimal residual risk is:

$$
\begin{aligned}
\omega^* &= \frac{\text{IR}}{2\lambda_R} \\
&= \frac{0.60}{2 \times 0.12} \\
&= 2.5\%
\end{aligned}
$$

The optimal value added is (see eq. (5.12)):

$$
\begin{aligned}
\mathrm{VA}^* &= \frac{\mathrm{IR}^2}{4\lambda_R} \\
&= \frac{0.60^2}{4 \times 0.12} \\
&= 0.75\%
\end{aligned}
$$

Problem 5.5. Oops. In fact, your information ratio is really only IR = 0.30. How much value added have you lost by setting your residual risk level according to Problem 4 instead of at its correct optimal level?

Solution. The optimal residual risk for IR = 0.30 is

$$
\begin{aligned}
\omega^* &= \frac{\mathrm{IR}}{2\lambda_R} \\
&= \frac{0.30}{2 \times 0.12} \\
&= 1.25\%
\end{aligned}
$$

The value added using the optimal residual risk is

$$
\begin{aligned}
\mathrm{VA}^* &= \frac{\mathrm{IR}^2}{4\lambda_R} \\
&= \frac{0.30^2}{4 \times 0.12} \\
&= 0.1875\%
\end{aligned}
$$

The value added using the residual risk from problem 4 is (see eq. (5.9)

$$
\begin{aligned}
\mathrm{VA}[\omega] &= \omega \cdot \mathrm{IR} - \lambda_R \cdot \omega^2 \\
&= 2.5\% \times 0.30 - 0.12 \times (2.5\%)^2 \\
&= 0
\end{aligned}
$$

Hence, by using the non-optimal residual risk from problem 4, we loose 0.1875% value added.

Problem 5.6. You are an active manager with an information ratio of IR = 0.50 (top quartile) an a target level of residual risk of 4 percent. What residual risk aversion should lead to that level of risk?

Solution. From eq. (5.11) we have

$$
\begin{aligned}
\lambda_R &= \frac{\mathrm{IR}}{2\omega^*} \\
&= \frac{0.50}{2 \times 4\%} \\
&= 0.0625/\%
\end{aligned}
$$

Chapter 5 technical appendix

Problem 5a.1. Demonstrate that

$$\left(\frac{f_Q}{\sigma_Q}\right)^2 = \left(\frac{f_B}{\sigma_B}\right)^2 + \mathrm{IR}^2$$

Solution. We start with the right-hand side and expand it using (5A.14).

$$\begin{aligned}
\left(\frac{f_B}{\sigma_B}\right)^2 + \mathrm{IR}^2 &= \left(\frac{f_B}{\sigma_B}\right)^2 + \left(\frac{f_Q}{\sigma_Q}\right)^2 \left(\frac{\omega_Q}{\sigma_Q}\right)^2 \\
&= \frac{f_B^2}{\sigma_B^2} + \frac{f_Q^2}{\sigma_Q^2} \frac{\sigma_Q^2 - \beta_Q^2 \sigma_B^2}{\sigma_Q^2} \\
&= \frac{f_B^2}{\sigma_B^2} + \frac{f_Q^2}{\sigma_Q^2} - \beta_Q^2 \frac{f_Q^2 \sigma_B^2}{\sigma_Q^4} \\
&= \frac{f_B^2}{\sigma_B^2} + \frac{f_Q^2}{\sigma_Q^2} - \frac{f_B^2 \sigma_Q^4}{f_Q^2 \sigma_B^4} \frac{f_Q^2 \sigma_B^2}{\sigma_Q^4} \qquad (2A.37) \\
&= \frac{f_B^2}{\sigma_B^2} + \frac{f_Q^2}{\sigma_Q^2} - \frac{f_B^2}{\sigma_B^2} \\
&= \frac{f_Q^2}{\sigma_Q^2}.
\end{aligned}$$

Problem 5a.2. Demonstrate that

$$\beta_Q = \frac{\beta_C \cdot f_B}{\beta_C \cdot f_B + \alpha_C}$$

Note that $\beta_C = (\sigma_C/\sigma_B)^2$. In the absence of benchmark timing, i.e., if $f_B = \mu_B$, the alpha of portfolio C is the key to determining the beta of portfolio Q.

Solution. This follows straightforwardly from (2A.38):

$$\begin{aligned}
\beta_Q &= \frac{\beta_C \cdot f_B}{f_C} \\
&= \frac{\beta_C \cdot f_B}{\beta_C \cdot f_B + \alpha_C}.
\end{aligned}$$

Chapter 6

Problem 6.1. Manager A is a stock picker. He follows 250 companies, making new forecasts each quarter. His forecasts are 2 percent correlated with subsequent residual returns. Manager B engages in tacit asset allocation, timing four equity styles (value, growth, large, small) every quarter. (a) What must Manager B's skill level be to match Manager A's information ratio? (b) What information ratio could a sponsor achieve by employing both managers, assuming that Manager B has a skill level of 8 percent?

Solution.

(a) Manager A has an information ratio of

$$\begin{aligned} \text{IR} &= \text{IC}\sqrt{\text{BR}} \\ &= 0.02 \times \sqrt{1000} \\ &= 0.632 \end{aligned}$$

For manager B to have an information ratio of 0.632, his information coefficient would need to be

$$\begin{aligned} \text{IC} &= \text{IR}/\sqrt{\text{BR}} \\ &= 0.632/\sqrt{16} \\ &= 0.158 \end{aligned}$$

So manager B's forecasts would need to be 16% correlated with the subsequent residual returns.

(b) A sponsor could achieve an information ratio of

$$\begin{aligned} \text{IR} &= \sqrt{\text{IR}_\text{A}{}^2 + \text{IR}_\text{B}2} \\ &= \sqrt{0.632^2 + (0.08 * \sqrt{16})^2} \\ &= 0.71 \end{aligned}$$

if manager A and B's forecasts are independent and if manager B has an information coefficient of 0.08 (a skill of 8%), giving him an IC of 0.32.

Problem 6.2. A stock picker follows 500 stocks and updates his alphas every month. He has an IC = 0.05 and an IR = 1.0. (a) How many bets does he make per year? (b) How many independent bets does he make per year? (c) What does this tell you about his alphas?

Solution.

(a) The stock picker makes $500 \times 12 = 6000$ bets per year.

(b) The stock pickers breadth is

$$\begin{aligned} \text{BR} &= \text{IR}^2/\text{IC}^2 \\ &= (1/.05)^2 \\ &= 400 \end{aligned}$$

so he makes 400 independent bets per year.

(c) Since the number of bets he makes per year is not equal to the number of independent bets he makes per year, his alphas are not independent.

Problem 6.3. In the example involving residual returns θ_n composed of 300 elements $\theta_{n,j}$, an investment manager must choose between three research programs:

(a) Follow 200 stocks each quarter and accurately forecast $\theta_{n,12}$ and $\theta_{n,15}$

(b) Follow 200 stocks each quarter and accurately forecast $\theta_{n,5}$ and $\theta_{n,105}$

(c) Follow 100 stocks each quarter and accurately forecast $\theta_{n,5}$, $\theta_{n,12}$, and $\theta_{n,105}$

Compare the three programs, all assumed to be equally costly. Which would be most effective (highest value added)?

Solution. For (a) and (b) there are 800 pieces of information each year, while for (c) there are only 400 pieces of information each year. Furthermore, the residual return of stock n is given by $\theta_n = \sum_{j=1}^{300} \theta_{n,j}$.

(a) Here, $\theta_{n,12}$ and $\theta_{n,15}$ are perfectly correlated with θ_n while all others are uncorrelated. Hence, we have $\text{STD}\{\theta_n\} = 17.32$ (see p 152) and $\text{STD}\{\theta_{n,12} + \theta_{n,15}\} = \sqrt{(0-1)^2 + (0-1)^2} = \sqrt{2}$ since the mean of each $\theta_{n,j}$ is zero and the standard deviation is 1. Furthermore, the covariance between our predictions and the actual return will be 2 since $\theta_{n,12}$ and $\theta_{n,15}$ are forecast perfectly. The IC is then given by the correlation between the forecasts and the residual return as

$$\begin{aligned} \text{IC} &= \frac{\text{Cov}\{\theta_n, \theta_{n,12} + \theta_{n,15}\}}{\text{STD}\{\theta_n\} \times \text{STD}\{\theta_{n,12} + \theta_{n,15}\}} \\ &= 2/(17.32 \times \sqrt{2}) \\ &= 0.0817 \end{aligned}$$

The information ratio is then given by

$$\begin{aligned} \text{IR} &= \text{IC}\sqrt{\text{BR}} \\ &= 0.0817 \times \sqrt{800} \\ &= 2.31 \end{aligned}$$

(b) This research program will have the same IR as (a). The only difference are the elements that are forecast accurately, but the number of correct forecasts does not change

(c) Using the same reasoning as in (a), we find that

$$\begin{aligned} \text{IC} &= \frac{\text{Cov}\{\theta_n, \theta_{n,5} + \theta_{n,12} + \theta_{n,105}\}}{\text{STD}\{\theta_n\} \times \text{STD}\{\theta_{n,5} + \theta_{n,12} + \theta_{n,105}\}} \\ &= \frac{3}{\sqrt{300}\sqrt{3}} \\ &= 0.1 \end{aligned}$$

so

$$\begin{aligned} \text{IR} &= 0.1 \times \sqrt{400} \\ &= 2 \end{aligned}$$

Hence, even though the skill of research program (c) would be better, there aren't enough bets made for the IR to be better than research programs (a) or (b). (a) and (b) will be the most effective strategies and should have the highest value added since $\text{VA}^2 \propto \text{IR}^2$.

Chapter 6 technical appendix

For the following exercises, consider the following model of a stock picker's forecast monthly alphas:

$$\alpha_n = a \cdot \theta_n + b \cdot z_n$$
$$\operatorname{Std}\{\alpha_n\} = \mathrm{IC} \cdot \operatorname{Std}\{\theta_n\} = \frac{\mathrm{IC} \cdot \omega_n}{\sqrt{12}}$$

where α_n is the forecast residual return, θ_n is the subsequent realized return, and z_n is a random variable with mean 0 and standard deviation 1, uncorrelated with θ_n and with z_m $(m \neq n)$.

Problem 6a.1. Given that $a = \mathrm{IC}^2$, what coefficient b will ensure that

$$\operatorname{Std}\{\alpha_n\} = \mathrm{IC} \cdot \operatorname{Std}\{\theta_n\} = \frac{\mathrm{IC} \cdot \omega_n}{\sqrt{12}}.$$

Solution. The phrasing of this question is liable to confuse. We are being asked to solve for b given that $a = \mathrm{IC}^2$ along with the other constraints of the model.

We proceed by computing $\operatorname{Var}\{\alpha_n\}$ and then substituting this into the equation $\operatorname{Var}\{\alpha_n\} = \mathrm{IC}^2 \cdot \operatorname{Var}\{\theta_n\}$.

$$\begin{aligned}\operatorname{Var}\{\alpha_n\} &= \operatorname{Var}\{a \cdot \theta_n + b \cdot z_n\} \\ &= a^2 \operatorname{Var}\{\theta_n\} + b^2 && \because \operatorname{Cov}\{\theta_n, z_n\} = 0,\ \operatorname{Var}\{z_n\} = 1.\end{aligned}$$

Hence, we obtain that

$$\begin{aligned} a^2 \operatorname{Var}\{\theta_n\} + b^2 &= \mathrm{IC}^2 \cdot \operatorname{Var}\{\theta_n\} \\ \therefore\ b^2 &= (\mathrm{IC}^2 - a^2)\operatorname{Var}\{\theta_n\} \\ \therefore\ b &= \sqrt{\mathrm{IC}^2 - a^2}\, \operatorname{Std}\{\theta_n\} \\ &= \sqrt{\frac{\mathrm{IC}^2 - a^2}{12}}\, \omega_n \end{aligned}$$

Note that if we assume that z_n is symmetrically distributed about 0, then it does not matter whether we choose the positive or the negative square root.

Then, we finally substitute in that $a = \mathrm{IC}^2$, obtaining that

$$b = \sqrt{\frac{\mathrm{IC}^2 - \mathrm{IC}^4}{12}}\, \omega_n.$$

Problem 6a.2. What is the manager's information coefficient in this model?

Solution. We apply the definition of the information coefficient and expand:

$$\begin{aligned} \mathrm{IC} &= \operatorname{Corr}\{\alpha, \theta_n\} \\ &= \frac{\operatorname{Cov}\{\alpha_n, \theta_n\}}{\operatorname{Std}\{\alpha_n\} \operatorname{Std}\{\theta_n\}} \\ &= \frac{a \operatorname{Var}\{\theta_n\}}{\mathrm{IC} \cdot \operatorname{Std}\{\theta_n\} \cdot \operatorname{Std}\{\theta_n\}} \\ &= \frac{a}{\mathrm{IC}}. \end{aligned}$$

This implies that

$$\mathrm{IC}^2 = a,$$

and so $\mathrm{IC} = \sqrt{a}$. Note that we must take the positive square root, since $\operatorname{Std}\{\alpha_n\} = \mathrm{IC} \cdot \operatorname{Std}\{\theta_n\}$, and standard deviations are always positive.

Problem 6a.3. Assume that the model applies to the 500 stocks in the S&P 500, with $a = 0.0001$ and $\omega_n = 20$ percent. What is the information ratio of the model, according to the fundamental law?

Solution. The fundamental law states that

$$\mathrm{IR} = \mathrm{IC} \cdot \sqrt{\mathrm{BR}}.$$

Since there are 500 stocks with alphas forecast monthly, we have that $\mathrm{BR} = 12 \times 500 = 6{,}000$. Then, by 6a.2, we have that $\mathrm{IC} = \sqrt{0.0001} = 0.01$. Hence

$$\mathrm{IR} = 0.01 \times \sqrt{6{,}000} = 0.775.$$

Problem 6a.4. Distinguish this model of alpha from the binary model introduced in the main part of the chapter.

Solution. First we recall the binary model of alpha introduced in the main part of the chapter. Here we have that

$$\theta_n = \sum_{j=1}^{m} \theta_{n,j},$$

where $\theta_{n,j}$ are random variables with mean 0 and standard deviation 1, and m is the number of these variables, which gives the number of components of the residual return θ_n. Our forecasting procedure gives us the value of $\theta_{n,1}$, but leaves us in the dark about the values of $\theta_{n,j}$ for $j > 1$.

To compare the two models, we may write

$$\alpha_n = a \cdot \theta_n + b \cdot z_n \tag{1}$$

$$\theta_{n,1} = \theta_n - \sum_{j=2}^{m} \theta_{n,j}, \tag{2}$$

since α_n and $\theta_{n,1}$ are the forecast alphas for the respective models.

Hence, one can see that the two models are different in several ways. The coefficient of θ_n in (2) is 1, whereas the coefficient of θ_n in (1) is a, which is not necessarily equal to 1. Likewise, $b \cdot z_n$ is a random variable with mean zero and standard deviation b, whereas $\sum_{j=2}^{m} \theta_{n,j}$ is a random variable with mean zero and standard deviation $\sqrt{m-1}$. We cannot simply set $a = 1$ and $b = \sqrt{m-1}$ to make the two models identical, since in the current model we also have the constraint $\mathrm{Std}\{\alpha_n\} = \mathrm{IC} \cdot \mathrm{Std}\{\theta_n\}$ to be mindful of. We know from 6a.2 that this constraint implies $\mathrm{IC}^2 = a$, and so $\mathrm{IC} = 1$ if $a = 1$. On the other hand, in the binary model we have $\mathrm{IC} = \frac{1}{\sqrt{m}}$, following the reasoning in the main body of the chapter. Hence, the two models are distinct unless $m = 1$ and $b = 0$, in which case the forecast alphas are perfect.

To summarise, each model gives the forecast alpha as a linear combination of the subsequent realised alpha and some other random variable. But these linear combinations are different in each model, as are the additional random variables. By adjusting the coefficients, one can make the two models give the forecast alpha as the same linear combinations of the realised alpha and another random variable, but then the information coefficients given by the models will certainly be distinct, due to the constraint that $\mathrm{Std}\{\alpha_n\} = \mathrm{IC} \cdot \mathrm{Std}\{\theta_n\}$ in the current model.

Chapter 7

Problem 7.1. According to the APT, what are the expected values of the u_n in Eq. (7.1)? What is the corresponding relationship for the CAPM?

Solution. According to the APT, the expected excess return is

$$\begin{aligned} f_n &= E\{r_n\} \\ &= E\left\{\sum_{k=1}^{K} X_{n,k} \cdot b_k + u_n\right\} \\ &= \sum_{k=1}^{K} X_{n,k} \cdot m_k \end{aligned}$$

where b_k is the factor return for factor k and m_k is the factor forecast for factor k. Hence the expected value of u_n is zero. This is in line with the CAPM which is a one factor APT model where the factor is the stock's beta:

$$\begin{aligned} f_n &= E\{r_n\} \\ &= E\left\{\beta_n r_M + \theta_n\right\} \\ &= \beta_n f_m \end{aligned}$$

where the expected residual return θ_n is zero.

Problem 7.2. Work by Fama and French, and others, over the past decade has identified size and book-to-price ratios as two critical factors determining expected returns. How would you build an APT model based on those two factors? Would the model require additional factors?

Solution. I would use one of the structural models presented in this chapter, and it seems like Structural Model 3 would be the most appropriate. The process might look something like:

1. Take a broad universe of stocks. For each year of historical returns, calculate the size and book to price ratio of each stock. The size will likely need to be standardized (since it is extensive), but I think the book to price ratio will be fine as is, since it is a ratio (it is intensive). This will determine the factor exposures

2. Regress the yearly returns against the size and book to price ratio of the stocks from step 1 and look for statistically significant correlations. This will give estimates for the factor returns.

3. Estimate (or calculate) the factor exposures for each stock for the current year we are trying to forecast. From these factor exposures and the historical returns, we can forecast the expected returns for the upcoming year

The model should not *require* any additional factors, but they might be useful for building better forecasts.

Problem 7.3. In the example shown in Table 7.2, most of the CAPM forecasts exceed the APT forecasts. Why? Are APT forecasts required to match CAPM forecasts of average?

Solution. The CAPM forecasts exceed the APT forecast because of the factor forecasts. In particular, the stocks in Table 7.2 tend to have above average size, as can be seen from the fact that the size exposures are above zero, when the size factor is forecast -1.5 percent. Similarly, the stocks in Table 7.2 tend to have below average growth, as can be seen from the fact that the growth exposures are negative, whilst the growth factor is forecast 2 percent.

The APT forecasts are not *required* to match the CAPM forecasts on average *per se*, but if the two sets of forecasts are to be consistent with each other, then the APT forecasts should match the CAPM forecasts on average. For instance, there will exist a different set of stocks to those in Table 7.2 where the companies have below average size and above average growth, which will cause the APT forecasts to exceed the CAPM forecasts. The factor forecasts should average out to give the market forecast.

Problem 7.4. In an earnings-to-price tilt fund, the portfolio holdings consist (approximately) of the benchmark plus a multiple c times the earnings-to-price factor portfolio (which has unit exposure to earnings-to-price and zero exposure all other factors). Thus, the tilt fund manager has an active exposure c to earnings-to-price. If the manager uses a constant multiple c over time, what does that imply about the manager's factor forecasts for earnings-to-price?

Solution. If a manager uses a constant c over time, it implies that his forecasts for earnings-to-price are not changing. However, the exposures to earnings to price will be changing, leading to changes in the stock forecasts.

Problem 7.5. You have built an APT model based on industry, growth, bond beta, size, and return on equity (ROE). This month your factor forecasts are

Heavy electrical industry	6.0%
Growth	2.0%
Bond beta	-1.0%
Size	-0.5%
ROE	1.0%

These forecasts lead to a benchmark expected excess return of 6.0 percent. Given the following data for GE,

Industry	Heavy electrical
Growth	-0.24
Bond beta	0.13
Size	1.56
ROE	0.15
Beta	1.10

what is its alpha according to your model

Solution. We can calculate the expected excess return as

$$f_{GE} = \sum_k X_k b_k$$

where the factor returns b_k are given in the first table and the factor exposures, X_k are given in the second table. Hence we have

$$\begin{aligned} f_{GE} &= 1 \times 6\% + (-0.24) \times 2.0\% + 0.13 \times (-1.0\%) + 1.56 \times (-0.5\%) + 0.15 \times (1.0\%) \\ &= 4.76\% \end{aligned}$$

Hence, alpha is given by

$$\begin{aligned} \alpha_{GE} &= f_{GE} - \beta_{GE} \times f_M \\ &= 4.76\% - 1.1 \times 6\% \\ &= -1.84\% \end{aligned}$$

Chapter 7 technical appendix

Problem 7a.1. A factor model contains an intercept if some weighted combination of the columns of $\mathbf{X}$ is equal to a vector of 1s. This will, of course, be true if one of the columns of $\mathbf{X}$ is a column of 1s. It will also be true if $\mathbf{X}$ contains a classification of stocks by industry or economic sector. The technical requirement for a model to have an intercept is that there exists a K-element vector g such that $\mathbf{e} = \mathbf{X} \cdot \mathbf{g}$. Assume that the model contains an intercept, and demonstrate that we can then determine the fraction of the portfolio invested in risky assets by looking only at the portfolio's factor exposures.

Solution. Let P be our portfolio. Then the factor exposures of the portfolio are given by $\mathbf{x}_P^T = \mathbf{h}_P^T \cdot \mathbf{X}$. We are assuming that the factor model has an intercept, so that there is a K-element vector $\mathbf{g}$ such that $\mathbf{e} = \mathbf{X} \cdot \mathbf{g}$. Hence we may calculate

$$\begin{aligned} \mathbf{x}_P^T \cdot \mathbf{g} &= \mathbf{h}_P^T \cdot \mathbf{X} \cdot \mathbf{g} \\ &= \mathbf{h}_P^T \cdot \mathbf{e}, \end{aligned}$$

which is the fraction of the portfolio P invested in risky assets.

Problem 7a.2. Show that a model that does not contain an intercept is indeed strange. In particular, show there will be a fully invested portfolio with zero exposures to all the factors—a portfolio P with $\mathbf{h}_P^T \cdot \mathbf{e} = 1$ (fully invested) and $\mathbf{x}_P = \mathbf{X}^T \cdot \mathbf{h}_P = \mathbf{0}$ (zero exposure to each factor).

Solution. We use tools from linear algebra. Suppose that the factor model does not contain an intercept. We have that $\mathbf{X}$ is a linear map from $\mathbb{R}^K \to \mathbb{R}^N$. Let $W = \operatorname{im} \mathbf{X}$ be the image of this map or, equivalently, the column space of the matrix $\mathbf{X}$. Since the factor model has no intercept, we know that $\mathbf{e} \notin W$. Hence $\dim W < N$, and so the orthogonal space $W^\perp$ is non-zero. Since $(W^\perp)^\perp = W$ and $\mathbf{e} \notin W$, then there exists a vector $\mathbf{h}_{P'} \in W^\perp$ such that $\mathbf{h}_{P'}^T \cdot \mathbf{e} =: e_{P'} \neq 0$. Moreover, since $\mathbf{h}_{P'} \in W^\perp$, we have that $\mathbf{X}^T \cdot \mathbf{h}_{P'} = \mathbf{0}$. Then we can let $\mathbf{h}_P = \frac{1}{e_{P'}} \mathbf{h}_{P'}$. By construction, we then have $\mathbf{h}_P^T \cdot \mathbf{e} = e_{P'}/e_{P'} = 1$ and $\mathbf{x}_P = \mathbf{X}^T \cdot \mathbf{h}_P = \frac{1}{e_{P'}} \cdot \mathbf{0} = \mathbf{0}$. Hence $\mathbf{h}_P$ gives the holdings of a fully invested portfolio with zero exposures to all the factors.

Note that the converse of this problem also holds. That is, if there is a fully invested portfolio which has zero exposures to all the factors, then the model cannot have an intercept. This can be reasoned into a similar way to Problem 7a.1. Namely, let P be the fully invested portfolio with zero exposures to all the factors and let $\mathbf{g}$ be the intercept of the model. Then

$$\begin{aligned} 1 &= \mathbf{h}_P^T \cdot \mathbf{e} \\ &= \mathbf{h}_P^T \cdot \mathbf{X} \cdot \mathbf{g} \\ &= \mathbf{0}^T \cdot \mathbf{g} \\ &= 0, \end{aligned}$$

which is clearly a contradiction. Hence there can be no model with an intercept where there exists a fully invested portfolio which has zero exposure to all factors.

Chapter 8

Problem 8.1. In the simple stock example described in the text, value a European call option on the stock with a strike price of 50, maturing at the end of the 1-month period. The option cash flows at the end of the period are $\text{Max}\{0, p(t,s) - 50\}$, where $p(t,s)$ is the stock price at time t in state s.

Solution. In the stock example in the text, a stock is currently valued at 50 and in 1 month will be worth either 49 ($p_{down} = 49$) or 53 ($p_{up} = 53$) with equal probability($\pi_{up} = \pi_{down} = 0.5$). The risk free interest rate, i_F over the year is 6 percent so that the return after 1 month is given by $R_F = (1 + i_F)^{1/12}) = 1.00487$. Furthermore, the valuation multiples are $\nu_{up} = 0.62$ and $\nu_{down} = 1.38$. We can use equations 8.8 and 8.9 to value the stock as

$$\begin{aligned} p_0 &= \frac{\pi_{up}\nu_{up}p_{up} + \pi_{down}\nu_{down}p_{down}}{R_F} \\ &= \frac{0.5 \times 0.62 \times 53 + 0.5 \times 1.38 \times 49}{1.00487} \\ &= 50 \end{aligned}$$

The value of the option can be calculated similarly by replacing the stock price at the end of the period with the value of the option at the end of the period. The value of the option is either 0 or 3, if the stock went down or up respectively. Hence, the current value of the option is

$$\begin{aligned} p_0 &= \frac{\pi_{up}\nu_{up}c_{up} + \pi_{down}\nu_{down}c_{down}}{R_F} \\ &= \frac{0.5 \times 0.62 \times 3 + 0.5 \times 1.38 \times 0}{1.00487} \\ &= 0.93 \end{aligned}$$

Problem 8.2. Compare Eq. (8.16) to the CAPM result for expected returns, to relate ν to r_Q. Impose the requirement that $E\{\nu\} = 1$ to determine ν exactly as a function of r_Q.

Solution. Equation 8.16 says that the expected return is given by

$$\text{E}\{R\} = 1 + i_F - \text{Cov}\{\nu, R\}$$

By comparing to the expected return according to the CAPM

$$\text{E}\{R\} = 1 + i_F + \beta f_Q$$

we find that

$$\begin{aligned} \text{Cov}\{\nu, R\} &= -\beta f_Q \\ &= -\frac{\text{Cov}\{r_Q, R\}}{\sigma_Q^2} f_Q \end{aligned}$$

Using the definition of covariance,

$$\begin{aligned} \text{E}\{\nu \cdot R\} - \text{E}\{\nu\}\text{E}\{R\} &= -\frac{\text{E}\{r_Q \cdot R\} - \text{E}\{r_Q\}\text{E}\{R\}}{\sigma_Q^2} f_Q \\ \text{E}\{\nu \cdot R\} &= \text{E}\{\nu\}\text{E}\{R\} - \frac{\text{E}\{r_Q \cdot R\} - f_Q\text{E}\{R\}}{\sigma_Q^2} f_Q \\ \text{E}\{\nu \cdot R\} &= \text{E}\left\{R\left[\text{E}\{\nu\} + \frac{f_Q}{\sigma_Q^2}(f_Q - r_Q)\right]\right\} \end{aligned}$$

Which implies

$$\nu = 1 + \frac{f_Q}{\sigma_Q^2}(f_Q - r_Q)$$

after imposing the condition that $\mathrm{E}\{\nu\} = 1$.

Problem 8.3. Using the simple stock example in the text, (a) price an instrument which pays \$1 in state 1 [$\mathrm{cf}(t,1) = 1$] and -\$1 in state 2 [$\mathrm{cf}(t,2) = -1$]. (b) What is the expected return to this asset? (c) What is its beta with respect to the stock? (d) How does this relate to the breakdown of Eq. (8.7)?

Solution.

(a) State 1 corresponds to when the stock is down and state two corresponds to when the stock is up. Using the same procedure and valuation multiples as in problem 1, the price is

$$\begin{aligned} p_0 &= \frac{0.5}{1.00487}(\times 1.38 \times 1 - 0.62 \times 1) \\ &= 0.378 \end{aligned}$$

(b) The expected return is

$$\begin{aligned} \mathrm{E}\{\mathrm{R}\} &= 0.5 \times 1 + 0.5 \times -1 \\ &= 0 \end{aligned}$$

(c) The beta of the asset (A) with respect to the stock (S) is

$$\begin{aligned} \beta &= \frac{\mathrm{Cov}\{A, S\}}{\sigma_S^2} \\ &= \frac{(49-51) \times (1-0)/2 + (53-51) \times (-1-0)/2}{(49-51)^2/2 + (53-51)^2/2} \\ &= \frac{-2}{4} \\ &= -1/2 \end{aligned}$$

(d) According to equation 8.7

$$\begin{aligned} p_0 &= \frac{\mathrm{E}\{\mathrm{cf}\}}{1 + i_F + \beta f_S} \\ &= 0 \end{aligned}$$

Because the expected value of the asset is zero, the price will always be zero, an equation (8.7) will therefore not be able to properly value the stock, regardless of the value of the discount rate in the denominator.

Problem 8.4. You believe that stock X is 25 percent undervalued, and that it will take 3.1 years for half of this misvaluation to disappear. What is your forecast for the alpha of stock X over the next year?

Solution. We want to use equation (8.22) but first we have to define κ and γ. κ is given as 0.25 and γ can be found from $\tau = -0.69/\ln\{\gamma\}$ where $\tau = 3.1$ years is the misvaluation half life. Hence, $\gamma = \exp(-0.69/3.1) = 0.80$. Plugging these values into equation 8.22, we find

$$\begin{aligned} \alpha &= (1 + i_F) \cdot \left[\frac{\kappa \cdot (1-\gamma)}{1 + \kappa \cdot \gamma}\right] \\ &= (1.06) \cdot \left[\frac{0.25 \times (1-0.8)}{1 + 0.25 \times 0.8}\right] \\ &= 0.044 \end{aligned}$$

Chapter 8 technical appendix

Problem 8a.1. Using the definitions from the technical appendix to Chap. 2, what is the characteristic associated with portfolio S?

Solution. By (2A.3), the attribute of a portfolio with holdings $\mathbf{h}$ in risky assets and variance σ^2 is

$$\frac{\mathbf{Vh}}{\sigma^2}.$$

By (8A.35), the holdings of portfolio S in risky assets are

$$\mathbf{h}_S = \frac{-\mathrm{SR}_Q \cdot (1 + i_F)}{\sigma_Q(1 + \mathrm{SR}_Q^2)} \mathbf{h}_Q.$$

Hence

$$\begin{aligned} \sigma_S^2 &= \left(\frac{-\mathrm{SR}_Q \cdot (1 + i_F)}{\sigma_Q(1 + \mathrm{SR}_Q^2)} \right)^2 \sigma_Q^2 \\ &= \frac{\mathrm{SR}_Q^2 (1 + i_F)^2}{(1 + \mathrm{SR}_Q^2)^2}. \end{aligned}$$

We therefore calculate the attribute associated to portfolio S to be

$$\begin{aligned} \frac{\mathbf{Vh}_S}{\sigma_S^2} &= \frac{(1 + \mathrm{SR}_Q^2)^2}{\mathrm{SR}_Q^2 (1 + i_F)^2} \frac{-\mathrm{SR}_Q \cdot (1 + i_F)}{\sigma_Q(1 + \mathrm{SR}_Q^2)} \mathbf{Vh}_Q \\ &= \frac{-(1 + \mathrm{SR}_Q^2)}{\sigma_Q \mathrm{SR}_Q (1 + i_F)} \mathbf{Vh}_Q \\ &= \frac{-(1 + \mathrm{SR}_Q^2)}{\sigma_Q \mathrm{SR}_Q (1 + i_F)} \frac{\sigma_Q^2}{f_Q} \mathbf{f} && \because (2A.36) \\ &= \frac{-(1 + \mathrm{SR}_Q^2)}{\mathrm{SR}_Q^2 (1 + i_F)} \mathbf{f}. \end{aligned}$$

Problem 8a.2. Show that the portfolio S holdings in risky assets satisfy

$$\mathbf{V} \cdot \mathbf{h}_S = -E\left\{R_S\right\} \cdot \mathbf{f}.$$

Solution. The holdings of portfolio S in risky assets are

$$\mathbf{h}_S = \frac{-\mathrm{SR}_Q \cdot (1 + i_F)}{\sigma_Q(1 + \mathrm{SR}_Q^2)} \mathbf{h}_Q.$$

Hence

$$\begin{aligned} \mathbf{Vh}_S &= \frac{-\mathrm{SR}_Q \cdot (1 + i_F)}{\sigma_Q(1 + \mathrm{SR}_Q^2)} \mathbf{Vh}_Q \\ &= \frac{-\mathrm{SR}_Q \cdot (1 + i_F)}{\sigma_Q(1 + \mathrm{SR}_Q^2)} \frac{\sigma_Q^2}{f_Q} \mathbf{f} \\ &= -\frac{1 + i_F}{1 + \mathrm{SR}_Q^2} \mathbf{f}. \end{aligned}$$

Now we show that the right-hand side of the equation in the problem is also equal to this.

$$\begin{aligned}
E\left\{R_S\right\} &= E\left\{R_F + \frac{-\mathrm{SR}_Q \cdot (1+i_F)}{\sigma_Q(1+\mathrm{SR}_Q^2)}(R_Q - R_F)\right\} \\
&= 1 + i_F - \frac{\mathrm{SR}_Q \cdot (1+i_F)}{\sigma_Q(1+\mathrm{SR}_Q^2)} f_Q \\
&= \frac{(1+i_F)(1+\mathrm{SR}_Q^2)}{1+\mathrm{SR}_Q^2} - \frac{\mathrm{SR}_Q^2 \cdot (1+i_F)}{1+\mathrm{SR}_Q^2} \\
&= \frac{1+i_F}{1+\mathrm{SR}_Q^2}.
\end{aligned}$$

Putting these two sets of calculations together yields the result.

Problem 8a.3. Show that portfolio S exists even if $f_C < 0$, and that if $f_C = 0$, then portfolio S will consist of 100 percent cash plus offsetting long and short positions in risky assets.

Solution. We start by proceeding similarly to the proof of Proposition 4. We consider a portfolio $P(w)$ composed of a fraction w invested in some arbitrary portfolio P, which we now do not assume to be fully invested, along with a fraction $(1-w)$ invested in portfolio F. Its total return is still

$$R_P(w) = R_F + w \cdot (R_P - R_F).$$

We again get that the optimal w is

$$w_P = \frac{-\mathrm{SR}_P \cdot (1+i_F)}{\sigma_P \cdot (1+\mathrm{SR}_P^2)}$$

with associated optimal expected second moment

$$E\left\{R_P^2(w_P)\right\} = \frac{(1+i_F)^2}{1+\mathrm{SR}_P^2}.$$

We again achieve the minimum second moment over all portfolios by maximising SR_P^2. We can maximise this by choosing $P = q$, the portfolio from Proposition 2 in the technical appendix from Chapter 2, which is not necessarily fully invested but exists even if $f_C < 0$. By (2A.29), we get that the exposure e_q of portfolio q is zero if $f_C = 0$. This means that portfolio q consists of cash plus offsetting long and short positions in risky assets. Since portfolio S consists of q along with cash, portfolio S also consists of cash plus offsetting long and short positions.

Problem 8a.4. Prove the portfolio S analog of Proposition 1 in the technical appendix of Chap. 7, i.e., that the factor model $(\mathbf{X}, \mathbf{F}, \Delta)$ explains expected excess returns if and only if portfolio S is diversified with respect to $(\mathbf{X}, \mathbf{F}, \Delta)$.

Solution. We first show that if portfolio S is diversified with respect to $(\mathbf{X}, \mathbf{F}, \Delta)$, then $(\mathbf{X}, \mathbf{F}, \Delta)$ explains expected excess returns. Portfolio S being diversified with respect to $(\mathbf{X}, \mathbf{F}, \Delta)$ means that S has minimal risk among all portfolios $\mathbf{h}$ with $\mathbf{X}^T \cdot \mathbf{h} = \mathbf{x}_S$. We proceed as in the proof of Proposition 1 in the technical appendix of Chapter 7 and consider the problem of minimising

$$\frac{\mathbf{h}^T \cdot \mathbf{V} \cdot \mathbf{h}}{2}$$

subject to

$$\mathbf{X}^T \cdot \mathbf{h} = \mathbf{x}_S.$$

We obtain the equation

$$\mathbf{V} \cdot \mathbf{h} = \mathbf{X} \cdot \boldsymbol{\pi},$$

where $\boldsymbol{\pi}$ is our vector of Lagrange multipliers. Since we are assuming that S is diversified with respect to $(\mathbf{X}, \mathbf{F}, \Delta)$, we know that $\mathbf{h}_S$ solves this equation, so that

$$\mathbf{V} \cdot \mathbf{h}_S = \mathbf{X} \cdot \boldsymbol{\pi}.$$

From Exercise 8a.2 above, we know that

$$\mathbf{V} \cdot \mathbf{h}_S = -E\left\{R_S\right\} \cdot \mathbf{f},$$

which implies that

$$\begin{aligned}\mathbf{f} &= \frac{-1}{E\left\{R_S\right\}} \mathbf{V} \cdot \mathbf{h}_S \\ &= \frac{-1}{E\left\{R_S\right\}} \mathbf{X} \cdot \boldsymbol{\pi}.\end{aligned}$$

Hence, if we let $\mathbf{m} = \frac{-1}{E\{R_S\}} \boldsymbol{\pi}$, then we obtain that $(\mathbf{X}, \mathbf{F}, \Delta)$ explains expected excess returns via this vector.

Now we show the converse implication, that if $(\mathbf{X}, \mathbf{F}, \Delta)$ explains expected excess returns, then portfolio S is diversified with respect to $(\mathbf{X}, \mathbf{F}, \Delta)$. To this end, we suppose for contradiction that $(\mathbf{X}, \mathbf{F}, \Delta)$ explains expected excess returns and that portfolio S is not diversified with respect to $(\mathbf{X}, \mathbf{F}, \Delta)$. This means that there exists a portfolio P with $\sigma_P^2 < \sigma_S^2$ such that

$$\mathbf{x}_S = \mathbf{X}^T \cdot \mathbf{h}_P = \mathbf{x}_P,$$

and that there exists a vector $\mathbf{m}$ such that

$$\mathbf{f} = \mathbf{X}\mathbf{m}.$$

We have that portfolio S minimises the second moment of total return, which we can expand as follows:

$$\begin{aligned}E\left\{R_S^2\right\} &= E\left\{(1 + i_F + r_S)^2\right\} \\ &= E\left\{1 + 2i_F r_S + i_F^2 + r_S^2 + 2i_F + 2r_S\right\} \\ &= 1 + i_F^2 + 2i_F + (2i_F + 2)E\left\{r_S\right\} + E\left\{r_S^2\right\} \\ &= 1 + i_F^2 + 2i_F + (2i_F + 2)E\left\{r_S\right\} + E\left\{r_S\right\}^2 + \operatorname{Var}\left\{r_S\right\}.\end{aligned}$$

We likewise have that

$$E\left\{R_P^2\right\} = 1 + i_F^2 + 2i_F + (2i_F + 2)E\left\{r_P\right\} + E\left\{r_P\right\}^2 + \operatorname{Var}\left\{r_P\right\}.$$

By using the vector $\mathbf{m}$ which explains expected excess returns, we deduce

$$\begin{aligned}E\left\{r_S\right\} &= \mathbf{f}^T \cdot h_S \\ &= \mathbf{m}^T \cdot \mathbf{X}^T \cdot h_S \\ &= \mathbf{m}^T \cdot \mathbf{x}_S \\ &= \mathbf{m}^T \cdot \mathbf{x}_P \\ &= \mathbf{m}^T \cdot \mathbf{X}^T \cdot h_P \\ &= \mathbf{f}^T \cdot h_P \\ &= E\left\{r_P\right\}.\end{aligned}$$

Then, since $\sigma_P^2 < \sigma_S^2$, we have $\operatorname{Var}\left\{r_P\right\} < \operatorname{Var}\left\{r_S\right\}$. Using this fact, combined with $E\left\{r_S\right\} = E\left\{r_P\right\}$, we conclude that $E\left\{R_P^2\right\} < E\left\{R_S^2\right\}$. But this contradicts portfolio S being the portfolio which minimises the second moment of total return. Hence, if $(\mathbf{X}, \mathbf{F}, \Delta)$ explains expected excess returns, then portfolio S must be diversified with respect to $(\mathbf{X}, \mathbf{F}, \Delta)$.

Chapter 9

Problem 9.1. According to Modigliani and Miller (and ignoring tax effects), how would the value of a firm change if (a) it borrowed money to repurchase outstanding common stock, greatly increasing its leverage? (b) What if it changed its payout ratio?

Solution. Modigliani and Miller demonstrated that (1) dividend policy only influences the scheduling of cash flows received by the share holder and that it does not affect the total value of the payments and (2) a firms financing policy does not affect the total value of the firm. Hence, the value of the firm will not be affected in either case (a) or (b). The value of a firm comes from its profitable activities and not dividend and financing policies.

Problem 9.2. Discuss the problem of growth forecasts in the context of the constant-growth dividend discount model [Eq. (9.5)]. How would you reconcile the growth forecasts with the implied growth forecasts for AT&T in Tables 9.1 and 9.2?

Solution. The authors mention that the raw growth forecasts can be unrealistic due to bias. Since the dividend discount model depends sensitively on the growth forecasts, it is important to have good forecasts. The implied growth rates, which assume the asset is fairly priced, can help adjust the growth forecasts to more realistic value. For instance, the raw growth forecast for AT&T in table 9.2 is -19.21% and the implied growth rate, given in table 9.1, is 6.26%. Using equation (9.22) results in a more modest AT&T growth forecast of 2.21%. Hence, the implied growth rates can be used to correct unrealistic growth forecasts.

Problem 9.3. Stock X has a beta of 1.20 and pays no dividend. If the risk-free rate is 6 percent and the expected excess market return is 6 percent, what is stock X's implied growth rate?

Solution. Using equation (9.20), the implied growth rate is given by

$$\begin{aligned} g_X^* &= (i_F + \beta_X \cdot f_B) - \frac{d_X}{p_X} \\ &= 6\% + 1.2 \times 6\% - 0 \\ &= 13.2\% \end{aligned}$$

Problem 9.4. You are a manager who believes that book-to-price (B/P), earnings to price (E/P), and beta are the three variables that determine stock value. Given monthly B/P, E/P, and beta values for 500 stocks, how could you implement your strategy (a) using comparative valuation? (b) using returns-based analysis?

Solution.

(a) To use comparative valuation, we would regress the companies current price against the three variables to come up with a price equation in the form of (9.43). The error associated with our price function and the actual price would identify misvaluation. It would be wise to check for outliers to make sure that they are not dominating the regression coefficients and skewing the model.

(b) Given monthly attributes (or exposures) for each stock, we can regress an equation in the form of (9.46) to determine the factor returns, $b_k(t)$, during each time period (or for each month). We can then use these factor returns to model future returns given current exposures of each stock to the factors. It might also be useful to look at how the error, or idiosyncratic, terms vary with time. If they are constant in time, this would identify that our model can be improved by choosing appropriate factors.

Problem 9.5. A stock trading with a P/E ratio of 15 has a payout ratio of 0.5 and an expected return of 12 percent. What is its growth rate, according to the constant-growth DDM?

Solution. From equation (9.7), the growth rate is given by

$$g = i_F + f - \frac{d}{p}$$

The dividends are given by

$$d = \kappa \cdot e$$

where κ is the payout ratio and $e(t)$ are the earning. Since the P/E ratio is 15, we can write

$$\begin{aligned} \frac{d}{p} &= \kappa \cdot \frac{e}{p} \\ &= 0.5 \times \frac{1}{15} \\ &= 0.0\bar{3} \end{aligned}$$

Hence, given an expected return of 12 percent, the growth rate is

$$\begin{aligned} g &= 0.12 - 0.0\bar{3} \\ &= 0.08\bar{6} \end{aligned}$$

Chapter 9 technical appendix

Problem 9.1a. You forecast an alpha of 2 percent for stocks that have E/P above the benchmark average *and* IBES growth above the benchmark average. On average, what must your alpha forecasts be for stocks that do not satisfy these two criteria? If you assume an alpha of zero for stocks which have either above-average E/P or above-average IBES growth, but not both, what is your average alpha for stocks with E/P and IBES growth both below average?

Solution. Assume that the following groups all have equal capitalisation:

1. stocks with E/P above the benchmark average and IBES growth above the benchmark average,
2. stocks with E/P below the benchmark average and IBES growth above the benchmark average,
3. stocks with E/P above the benchmark average and IBES growth below the benchmark average,
4. stocks with E/P below the benchmark average and IBES growth below the benchmark average.

We forecast an alpha of 2 percent for stocks in group 1. If this forecast is weighted with the forecasts for the other three groups, it must balance out to zero. Hence, on average, for stocks in groups 2, 3, and 4 our alpha forecasts must be $-2/3$ percent, so that $(3 \times -2/3) + 2 = 0$.

If, furthermore, we assume an alpha of zero for stocks in groups 2 and 3, then our average alpha forecast for stocks in group 4 must be -2, so that $2 + 0 + 0 - 2 = 0$.

Chapter 10

Problem 10.1. Assume that residual returns are uncorrelated, and that we will use an optimizer to maximize risk-adjusted residual return. Using the data in Table 10.3, what asset will the optimizer choose as the largest positive active holding? How would that change if we had assigned $\alpha = 1$ for buys and $\alpha = -1$ for sells? *Hint:* At optimality, assuming uncorrelated residual returns, the optimal active holdings are

$$h_n = \frac{\alpha_n}{2\lambda_R \omega_n^2}$$

Solution. Using the alpha's from table 10.3 from the refined forecasts, the optimizer will chose the stock that maximizes h_n as the largest positive active holding. This means that only stocks with positive α need to be considered. If we calculate h_a for all stocks with a positive alpha (regardless of tolerance to risk, λ_R) we find that the α/ω_n^2 ratio is largest for AT&T, where it equals 0.056635. Even though AT&T has the smallest α, it also has the smallest ω. If instead of the refined forecasts, buy recommendations were given $\alpha = 1$ and sell recommendations were given $\alpha = -1$, the optimizer would just pick the buy recommendation with the smallest residual volatility, which in this case is AT&T.

Problem 10.2. For the situation described in Problem 1, show that using the forecasting rule of thumb, we assume equal risk for each asset. What happens if we just use $\alpha = 1$ for buys and $\alpha = -1$ for sells?

Solution. The forecasting rule of thumb states

$$\text{Refined forecast} = \text{volatility} \times \text{IC} \times \text{score}$$

If we assign just assign $\alpha = 1$ for buy and $\alpha = -1$ for sell, since the IC are constant and the scores are 1 for buy and 1 for sell, we find that the volatility for each stock is

$$\begin{aligned}\text{volatility} &= 1/(0.09 \times 1) \text{ for buy}\\ \text{volatility} &= -1/(0.09 \times -1) \text{ for sell}\end{aligned}$$

Hence, using $\alpha = 1$ for buys and $\alpha = -1$ for sells assumes equal risk for each asset.

Problem 10.3. Use the basic forecasting formula [Eq. (10.1)] to derive Eq. (10.20), the refined forecast in the case of one asset and two forecasts.

Solution. Since Eq (10.20) is a refined forecast, let us start with the definition of the refined forecast in Eq. (10.2). We have

$$\boldsymbol{\phi} = \text{Cov}\{\boldsymbol{r}, \boldsymbol{g}\} \cdot \text{Var}^{-1}\{\boldsymbol{g}\} \cdot (\boldsymbol{g} - E\{\boldsymbol{g}\})$$

For the case of one asset and two forecasts, the vectors and matrices can be written as

$$\begin{aligned}
\text{Cov}\{\boldsymbol{r}, \boldsymbol{g}\} &= \text{Std}\{r\} \cdot \boldsymbol{\rho_{r,g}} \cdot \text{Std}\{\boldsymbol{g}\}\\
&= \sigma_r \cdot \begin{bmatrix}\text{IC}_{g_1} & \text{IC}_{g_2}\end{bmatrix} \cdot \begin{bmatrix}\text{Std}\{g_1\} & 0\\ 0 & \text{Std}\{g_2\}\end{bmatrix}\\
\text{Var}^{-1}\{\boldsymbol{g}\} &= \text{Std}\{\boldsymbol{g}\}^{-1} \boldsymbol{\rho}^{-1}_{\boldsymbol{g_1,g_2}} \text{Std}\{\boldsymbol{g}\}^{-1}\\
&= \begin{bmatrix}\frac{1}{\text{Std}\{g_1\}} & 0\\ 0 & \frac{1}{\text{Std}\{g_2\}}\end{bmatrix} \frac{1}{\rho_{g_1g_1}\rho_{g_2g_2} - \rho_{g_1g_2}\rho_{g_2g_1}} \begin{bmatrix}\rho_{g_2g_2} & -\rho_{g_2g_1}\\ -\rho_{g_1g_2} & \rho_{g_1g_1}\end{bmatrix} \cdot \begin{bmatrix}\frac{1}{\text{Std}\{g_1\}} & 0\\ 0 & \frac{1}{\text{Std}\{g_2\}}\end{bmatrix}\\
&= \begin{bmatrix}\frac{1}{\text{Std}\{g_1\}} & 0\\ 0 & \frac{1}{\text{Std}\{g_2\}}\end{bmatrix} \frac{1}{1 - \rho^2_{g_1g_2}} \begin{bmatrix}1 & -\rho_{g_2g_1}\\ -\rho_{g_1g_2} & 1\end{bmatrix} \cdot \begin{bmatrix}\frac{1}{\text{Std}\{g_1\}} & 0\\ 0 & \frac{1}{\text{Std}\{g_2\}}\end{bmatrix}\\
\boldsymbol{g} - E\{\boldsymbol{g}\} &= \begin{bmatrix}g_1 - m_{g_1}\\ g_2 - m_{g_2}\end{bmatrix}
\end{aligned}$$

Here, we have used the usual definitions of the information coefficient and $g_{1(2)}$ and $m_{g_{1(2)}}$ is the forecast and mean of signal 1 (2) respectively. The ρ represent correlations and σ_r the standard deviation of the return. The inverse variance was calculated using the usual formula to invert a 2 by two matrix, and then simplified using the fact that self correlations (i.e. ρ_{11}) are equal to 1. Now that we have these expressions, we can determine $\boldsymbol{\phi}$ as

$$\boldsymbol{\phi} = \sigma_r \cdot \begin{bmatrix} \mathrm{IC}_{g_1} & \mathrm{IC}_{g_2} \end{bmatrix} \cdot \frac{1}{1-\rho_{g_1 g_2}^2} \begin{bmatrix} 1 & -\rho_{g_2 g_1} \\ -\rho_{g_1 g_2} & 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\mathrm{Std}\{g_1\}} & 0 \\ 0 & \frac{1}{\mathrm{Std}\{g_2\}} \end{bmatrix} \cdot \begin{bmatrix} g_1 - m_{g_1} \\ g_2 - m_{g_2} \end{bmatrix}$$

Multiplying the last two matrices give us the scores z_{g_1} and z_{g_2}. Multiplying the first two matrices give us

$$\begin{bmatrix} \mathrm{IC}_{g_1} & \mathrm{IC}_{g_2} \end{bmatrix} \cdot \frac{1}{1-\rho_{g_1 g_2}^2} \begin{bmatrix} 1 & -\rho_{g_2 g_1} \\ -\rho_{g_1 g_2} & 1 \end{bmatrix} = \frac{1}{1-\rho_{g_1 g_2}^2} \begin{bmatrix} \mathrm{IC}_{g_1} - \mathrm{IC}_{g_2}\rho_{g_1 g_2} & \mathrm{IC}_{g_2} - \mathrm{IC}_{g_1}\rho_{g_2 g_1} \end{bmatrix}$$

Multiplying the remaining matrices, we find

$$\begin{aligned} \boldsymbol{\phi} &= \sigma_r \frac{1}{1-\rho_{g_1 g_2}^2} \begin{bmatrix} \mathrm{IC}_{g_1} - \mathrm{IC}_{g_2}\rho_{g_1 g_2} & \mathrm{IC}_{g_2} - \mathrm{IC}_{g_1}\rho_{g_2 g_1} \end{bmatrix} \cdot \begin{bmatrix} z_{g_1} \\ z_{g_2} \end{bmatrix} \\ &= \sigma_r \frac{1}{1-\rho_{g_1 g_2}^2} \left([\mathrm{IC}_{g_1} - \mathrm{IC}_{g_2}\rho_{g_1 g_2}]\, z_{g_1} + [\mathrm{IC}_{g_2} - \mathrm{IC}_{g_1}\rho_{g_2 g_1}]\, z_{g_2} \right) \\ &= \sigma_r \left(IC_{g_1}^* z_{g_1} + IC_{g_1}^* z_{g_2} \right) \end{aligned}$$

where we have used the definitions of $\mathrm{IC}_{g_1}^*$ and $\mathrm{IC}_{g_2}^*$ from equations (10.21) and (10.22). This completes the derivation of Eq. (10.20).

Problem 10.4. In the case of two forecasts [Eq. (10.20)], (a) what is the variance of the combined forecast? (b) What is its covariance with the return? (c) Verify explicitly that the combination of g and g' in the example leads to an IC of 0.1090. Compare this to the result from Eq. (10.27).

Solution. The combined forecast is given by Eq. (10.20):

$$\phi = \mathrm{Std}\{r\} \cdot \mathrm{IC}_g^* \cdot z_g + \mathrm{Std}\{r\} \cdot \mathrm{IC}_{g'} \cdot z_{g'}.$$

We will write $\omega = \mathrm{Std}\{r\}$ and $\rho = \rho_{g,g'}$.

(a) Note that $\mathrm{Var}\{z_g\} = \mathrm{Var}\{z_{g'}\} = 1$, since these variables are normalised. Moreover,

$$\mathrm{Cov}\{z_g, z_{g'}\} = \frac{\mathrm{Cov}\{g, g'\}}{\mathrm{Std}\{g\}\,\mathrm{Std}\{g'\}} = \rho.$$

Hence,

$$\begin{aligned} \mathrm{Var}\{\phi\} &= \mathrm{Cov}\left\{\omega \cdot \mathrm{IC}_g^* \cdot z_g + \omega \cdot \mathrm{IC}_{g'}^* \cdot z_{g'},\, \omega \cdot \mathrm{IC}_g^* \cdot z_g + \omega \cdot \mathrm{IC}_{g'}^* \cdot z_{g'}\right\} \\ &= \omega^2((\mathrm{IC}_g^*)^2 + 2\rho \mathrm{IC}_g^* \mathrm{IC}_{g'}^* + (\mathrm{IC}_{g'}^*)^2). \end{aligned}$$

(b) Note that

$$\mathrm{Cov}\{r, z_g\} = \frac{\mathrm{Cov}\{r, g\}}{\mathrm{Std}\{g\}} = \omega \cdot \mathrm{IC}_g,$$

and similarly $\mathrm{Cov}\{r, z_{g'}\} = \omega \cdot \mathrm{IC}_{g'}$. The covariance of the combined forecast with the return is then

$$\begin{aligned} \mathrm{Cov}\{r, \phi\} &= \mathrm{Cov}\left\{r,\, \omega \cdot \mathrm{IC}_g^* \cdot z_g + \omega \cdot \mathrm{IC}_{g'} \cdot z_{g'}\right\} \\ &= \omega^2(\mathrm{IC}_g^* \mathrm{IC}_g + \mathrm{IC}_g^* \mathrm{IC}_g). \end{aligned}$$

(c) The IC of the combined forecast is equal to its correlation with the return

$$\begin{aligned} \mathrm{IC}_\phi &= \frac{\mathrm{Cov}\{r, \phi\}}{\mathrm{Std}\{r\}\,\mathrm{Std}\{\phi\}} \\ &= \frac{\omega^2(\mathrm{IC}_g^* \mathrm{IC}_g + \mathrm{IC}_g^* \mathrm{IC}_g)}{\omega\sqrt{\omega^2((\mathrm{IC}_g^*)^2 + 2\rho \mathrm{IC}_g^* \mathrm{IC}_{g'}^* + (\mathrm{IC}_{g'}^*)^2)}} \\ &= \frac{\mathrm{IC}_g^* \mathrm{IC}_g + \mathrm{IC}_{g'}^* \mathrm{IC}_{g'}}{\sqrt{(\mathrm{IC}_g^*)^2 + 2\rho \mathrm{IC}_g^* \mathrm{IC}_{g'}^* + (\mathrm{IC}_{g'}^*)^2}} \end{aligned}$$

For the example in the text, $\text{IC}_g = 0.0833$, $\text{IC}_{g'} = 0.089$, and $\rho = 1/4$, so

$$\begin{aligned}
\text{IC}_g^* &= \frac{\text{IC}_g - \rho \text{IC}_{g'}}{1-\rho^2} \\
&= \frac{0.0833 - 0.25 \times 0.089}{1 - 1/16} \\
&= 0.8648,
\end{aligned}$$

and

$$\begin{aligned}
\text{IC}_{g'}^* &= \frac{\text{IC}_{g'} - \rho \text{IC}_g}{1-\rho^2} \\
&= \frac{0.089 - 0.25 \times 0.0833}{1 - 1/16} \\
&= 0.092712,
\end{aligned}$$

Plugging these values in, we find

$$\begin{aligned}
\text{IC}_\phi &= \frac{0.8648 \times 0.0833 + 0.092712 \times 0.089}{\sqrt{0.08648^2 + 2 \times 0.25 \times 0.8648 \times 0.092712 + 0.092712^2}} \\
&= 0.1091
\end{aligned}$$

We can compare this to Eq. (10.27), which states

$$\begin{aligned}
\text{IC}_\phi &= \sqrt{\frac{\text{IC}_g^2 + \text{IC}_{g'}^2 - 2\rho \text{IC}_g \text{IC}_{g'}}{1-\rho^2}} \\
&= \sqrt{\frac{0.0833^2 + 0.089^2 - 2 \times 1/4 \times 0.0833 \times 0.089}{1 - 1/16}} \\
&= 0.1091
\end{aligned}$$

The difference is likely due to a rounding error in the calculation of the input terms, most likely the information coefficients.

Problem 10.5. You are using a neural net to forecast returns to one stock. The net inputs include fundamental counting data, analyst's forecasts, and past returns. The net combines these nonlinearly. How would the forecasting rule of thumb change under these circumstances?

Solution. The neural network will take the raw inputs and forecast the returns to the stock directly. Hence, it doesn't seem as if the rule of thumb [equation (10.11)] will apply since the conversion from raw signal to forecast is done behind the scenes. However, it should be straightforward to decompose the forecast of the neural network into the terms in the rule of thumb, since the volatility can be determined and a reasonable IC can be assigned.

Chapter 10 technical appendix

Problem 10.1a. Using Eq. (10A.21), what is the variance of the combined forecast? What is its covariance with the return? Remember that the combined forecast is simply a linear combination of signals. We know the volatilities and correlations of all the signals, and we know the correlation of each signal with the return.

Verify Eq. (10A.23) for the IC of the combined forecast. Demonstrate that when $K = 2$, it reduces to Eq. (10.27) in the main text of the chapter.

Solution. Eq. (10A.21) states that

$$\phi = \omega \cdot \mathbf{IC}_\mathbf{g}^T \cdot \boldsymbol{\rho}_\mathbf{g}^{-1} \cdot \mathbf{z}.$$

Note that $\text{Var}\{\mathbf{z}\} = \boldsymbol{\rho}_\mathbf{g}$ by Eq. (10A.18). Hence, the variance of the combined forecast is

$$\begin{aligned}\text{Var}\{\phi\} &= \omega^2 \cdot \mathbf{IC}_\mathbf{g}^T \cdot \boldsymbol{\rho}_\mathbf{g}^{-1} \cdot \text{Var}\{\mathbf{z}\} \cdot \boldsymbol{\rho}_\mathbf{g}^{-1} \cdot \mathbf{IC}_\mathbf{g} \\ &= \omega^2 \cdot \mathbf{IC}_\mathbf{g}^T \cdot \boldsymbol{\rho}_\mathbf{g}^{-1} \cdot \boldsymbol{\rho}_\mathbf{g} \cdot \boldsymbol{\rho}_\mathbf{g}^{-1} \cdot \mathbf{IC}_\mathbf{g} \\ &= \omega^2 \cdot \mathbf{IC}_\mathbf{g}^T \cdot \boldsymbol{\rho}_\mathbf{g}^{-1} \cdot \mathbf{IC}_\mathbf{g}.\end{aligned}$$

The covariance of the combined forecast with the return is

$$\begin{aligned}\text{Cov}\{\phi, r\} &= \text{Cov}\left\{\omega \cdot \mathbf{IC}_\mathbf{g}^T \cdot \boldsymbol{\rho}_\mathbf{g}^{-1} \cdot \mathbf{z}, r\right\} \\ &= \omega \cdot \mathbf{IC}_\mathbf{g}^T \cdot \boldsymbol{\rho}_\mathbf{g}^{-1} \cdot \text{Cov}\{\mathbf{z}, r\} \\ &= \omega \cdot \mathbf{IC}_\mathbf{g}^T \cdot \boldsymbol{\rho}_\mathbf{g}^{-1} \cdot \omega \cdot \mathbf{IC}_\mathbf{g} \\ &= \omega^2 \cdot \mathbf{IC}_\mathbf{g}^T \cdot \boldsymbol{\rho}_\mathbf{g}^{-1} \cdot \mathbf{IC}_\mathbf{g}.\end{aligned}$$

Therefore, the IC of the combined forecast is

$$\begin{aligned}\text{IC}_\phi &= \frac{\text{Cov}\{\phi, r\}}{\omega \cdot \text{Std}\{\phi\}} \\ &= \frac{\omega^2 \cdot \mathbf{IC}_\mathbf{g}^T \cdot \boldsymbol{\rho}_\mathbf{g}^{-1} \cdot \mathbf{IC}_\mathbf{g}}{\omega \cdot \sqrt{\omega^2 \cdot \mathbf{IC}_\mathbf{g}^T \cdot \boldsymbol{\rho}_\mathbf{g}^{-1} \cdot \mathbf{IC}_\mathbf{g}}} \\ &= \frac{\mathbf{IC}_\mathbf{g}^T \cdot \boldsymbol{\rho}_\mathbf{g}^{-1} \cdot \mathbf{IC}_\mathbf{g}}{\sqrt{\mathbf{IC}_\mathbf{g}^T \cdot \boldsymbol{\rho}_\mathbf{g}^{-1} \cdot \mathbf{IC}_\mathbf{g}}} \\ &= \sqrt{\mathbf{IC}_\mathbf{g}^T \cdot \boldsymbol{\rho}_\mathbf{g}^{-1} \cdot \mathbf{IC}_\mathbf{g}}.\end{aligned}$$

This verifies Eq. (10A.23). We now demonstrate that when $K = 2$, it reduces to Eq. (10.27) in the main text of the chapter. We let

$$\mathbf{g} = \begin{bmatrix} g \\ g' \end{bmatrix},$$

and $\rho = \text{Corr}\{g, g'\}$. Recall from Eq. (10A.25) that

$$\boldsymbol{\rho}_\mathbf{g}^{-1} = \frac{1}{1 - \rho^2} \begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix}.$$

This makes

$$\begin{aligned}\mathbf{IC}_\mathbf{g}^T \cdot \boldsymbol{\rho}_\mathbf{g}^{-1} \cdot \mathbf{IC}_\mathbf{g} &= \frac{1}{1 - \rho^2} \begin{bmatrix} \text{IC}_g & \text{IC}_{g'} \end{bmatrix} \begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix} \begin{bmatrix} \text{IC}_g \\ \text{IC}_{g'} \end{bmatrix} \\ &= \frac{1}{1 - \rho^2} (\text{IC}_g(\text{IC}_g - \rho \text{IC}_{g'}) + \text{IC}_{g'}(\text{IC}_{g'} - \rho \text{IC}_g)) \\ &= \frac{\text{IC}_g^2 - 2\rho \text{IC}_g \text{IC}_{g'} + \text{IC}_{g'}^2}{1 - \rho^2}.\end{aligned}$$

Hence, Eq. (10A.23) reduces to Eq. (10.27) from the main text of the chapter:

$$\mathrm{IC}_\phi = \sqrt{\frac{\mathrm{IC}_g^2 - 2\rho \mathrm{IC}_g \mathrm{IC}_{g'} + \mathrm{IC}_{g'}^2}{1-\rho^2}}.$$

Chapter 11

Problem 11.1. Signal 1 and Signal 2 have equal IC, and both exhibit signal volatilities proportional to asset volatilities. Do the two signals receive equal weight in the forecast of exceptional return?

Solution. Since both signals exhibit volatilities proportional to asset volatilities, we can use equation 11.14

$$\phi_n = \mathrm{IC} \cdot c_g \cdot z_{CS,n}$$

to determine the refined forecast of exceptional return. We see that there are two factors that weight the forecast, IC and c_g. We know the IC are the same, but the c_g can vary by signal. Hence, the signals do not necessilarly receive equal weight in the forecast of eceptional return.

Problem 11.2. What IR would you naïvely expect if you combined strategies A and C in Table 11.3? Why might the observed answer differ from the naïve result?

Solution. The naïve approach is to assume that the IR of the combined strategies is equal to $\sqrt{\mathrm{IR}_A^2 + \mathrm{IR}_B^2}$. However, this assumes that the strategies are uncorrelated. The actual IR, i.e. the IR associated with strategy B, will be lower than that predicted by the naïve approach due to correlation between the strategies.

Problem 11.3. How much should you shrink coefficient b, connecting raw signals and realized returns, estimated with $R^2 = 0.05$ after 120 months?

Solution. Assuming monthly observations, we determine the shrinkage as (see equation 11.31)

$$\begin{aligned}
\frac{b'}{b} &= \frac{1}{1 + 1/(T \cdot E\{R^2/(1-R^2)\})} \\
&= \frac{1}{1 + 1/(120 \times .0025/.9975)} \\
&= 0.2312
\end{aligned}$$

(see also table 11.4)

Chapter 12

Problem 12.1. What problems can arise in using scores instead of alphas in information analysis? Where in the analysis would these problems show up?

Solution. Scores tell about how the stock compares to others according to some criteria. For information analysis, we are interested in how these scores translate into returns so that we can evaluate the value in the information used to generate the scores. We can use scores to build portfolios, but we must then evaluate the performance of the portfolios to determine the value of the information. Note that most of the performance measures in this chapter (t-statistic, IC, IR) are all alpha dependent and do not depend directly on the scores. Hence, the problems in using scores instead of alphas would turn up in the performance evaluation step of the information analysis procedure.

Problem 12.2. What do you conclude from the information analysis presented concerning book-to-price ratios in the United States?

Solution. For all of the portfolios discussed, the t-statistic suggests that the results are not significant at the 95% confidence level. This suggests that the book to price ratio is not well suited to generate excess returns, a fact that makes sense in light of the fact that the book to price ratio is a common factor used to value stocks and so portfolios built on book to price should be relatively fairly valued. There is not much opportunity using this information.

Problem 12.3. Why might we see misleading results if we looked only at the relative performance of top- and bottom-quintile portfolios instead of looking at factor portfolio performance?

Solution. In this case, our factors are just alpha and beta. If we fail to analyze the returns in terms of these factors, we might come to the wrong conclusion about the information used to construct the portfolios. For example, if the returns on the lower and upper quintiles are 10% and 2% relative to the benchmark, it might be tempting to say that the information used to build the lower quintile portfolio is better (has a higher IC), however this may not be the case. If the beta of the lower quintile is 1.1 and the beta of the upper quintile is 1, the alphas would be 0 and 0.02 respectively . The upper quintile thus has a higher IC even though the total return is less than the lower quintile.

Problem 12.4. The probability of observing a $|t$ statistic $| > 2$, using random data, is only 5 percent. Hence our confidence in the estimate is 95 percent. Show that the probability of observing at least one $|t$ statistic $| > 2$ with 20 regressions on independent sets of random data is 64 percent.

Solution. The probability of observing at least one $|t$ statistic $| > 2$ is equal to 1 minus the probability of observing twenty $|t$ statistics $| < 2$. Hence,

$$\begin{aligned} P &= 1 - 0.95^{20} \\ &= 0.64 \end{aligned}$$

Problem 12.5. Show that the standard error of the information ratio is approximately $1/\sqrt{T}$, where T is the number of years of observation. Assume that you can measure the standard deviation of returns with perfect accuracy, so that all the error is in the estimate of the mean. Remember that the standard error of an estimated mean is $1/\sqrt{N}$, where N is the number of observations.

Solution. We can estimate the mean information ratio using data over a period of T years as

$$\bar{\mathrm{IR}} = \frac{1}{T} \sum_{t=1,T} \frac{\alpha_t}{\omega_t}$$

Assuming we can measure all α_t and ω_t with perfect accuracy, the standard error will simply be

$$\mathrm{SE}\{\bar{\mathrm{IR}}\} = \frac{\bar{\mathrm{IR}}}{\sqrt{T}}$$

since we have T samples.

Problem 12.6. You wish to analyze the value of corporate insider stock transactions. Should you analyze these using the standard cross-sectional methodology or an event study? If you use an event study, what conditioning variables will you consider?

Solution. An event study would be more appropriate for this type of analysis since the transactions will occur at different times for different companies. The information concerning these events will not arrive in the regular intervals that are needed for cross sectional analysis. Some conditioning variables to consider would be:

- Has there been a change in leadership? Insider stock transactions may signal the perception of the new leadership.
- Is the company planning to release or discontinue a product? Insider transactions may reveal how this move will be perceived by the public.
- Are insiders buying or selling? Could indicate perceived future value of the company.

Problem 12.7. Haugen and Baker (1996) have proposed an APT model in which expected factor returns are simply based on past 12-month moving averages. Applying this idea to the BARRA U.S. Equity model from January 1974 through March 1996 leads to an information ratio of 1.79. Applying this idea only to the risk indices in the model (using consensus expected returns for industries) leads to an information ratio of 1.26. (a) What information ratio would you expect to find from applying this model to industries only? (b) If the full application exhibits an information coefficient of 0.05, what is the implied breadth of the strategy?

Solution.

(a) When consensus expected returns are used for the industry factor returns, the information ratio decreases by 0.53. Since the full model is used to calculate the residual risk in both cases, this translates decrease in α by $0.53/\omega$. Hence, the industry factor returns in the model contribute 0.53 to the IR, assuming the full model is always applied to calculate the risk indices. When the model is applied only to industries, the IR will therefore be 0.53.

(b) We know that $\mathrm{IR} = \mathrm{IC}\sqrt{\mathrm{BR}}$. Hence, given an IR of 1.79 for the full model and an IC of 0.05, the breadth is

$$\begin{aligned}\mathrm{BR} &= \left(\frac{1.79}{0.05}\right)^2 \\ &= 1282\end{aligned}$$

Problem 12.8. A current get-rich-quick Web site guarantees that over the next 3 months, at least three stocks mentioned on the site will exhibit annualized returns of at least 300 percent. Assuming that all stock returns are independent, normally distributed, and with expected annual returns of 12 percent and risk of 35 percent, (a) what is the probability that over one quarter at least 3 stocks out of 500 exhibit annualized returns of at least 300%? (b) How many stocks must the Web site include for this probability to be 50 percent? (c) Identify at least two real-world deviations from the above assumptions, and discuss how they would affect the calculated probabilities.

Solution.

(a) For a given quarter, the risk is $\sigma = 0.35/\sqrt{4} = 0.175$. For a 3 month period, we are thus working with a normal distribution having a mean of 0.12 and a standard deviation 0.175. The probability that a single stock exhibits an annualized return of at least 300% in a given quarter (corresponding to a quarterly return of 75%) can be determined from the cumulative distribution function of the normal distribution as

$$\begin{aligned} P(x \geq 0.75) &= 1 - P(x < 0.75) \\ &= 1 - \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{0.75 - 0.12}{0.175\sqrt{2}}\right)\right] \\ &= 0.00016 \end{aligned}$$

The probability that at least 3 stocks out of 500 exhibit annualized returns of at least 300% is

$$\begin{aligned} P(y \geq 3) &= 1 - P(y = 0) - P(y = 1) - P(y = 2) \\ &= 1 - \binom{500}{0} P(x < 0.75)^{500} - \binom{500}{1} P(x < 0.75)^{499} P(x >= 0.75) \\ &\quad - \binom{500}{2} P(x < 0.75)^{499} P(x >= 0.75)^2 \\ &= 1 - 0.9235 - 0.07348 - .00292 \\ &= 0.786 \times 10^{-4} \end{aligned}$$

So the probability that at least three of 500 stocks will exhibit an annualized return of at least 300% in a given quarter is extreemly small.

(b) To find the number of stocks necessary so that the probability of at least three stocks exhibit annualized returns of 300% in a given quarter is 50 percent, we have to solve the equation

$$\begin{aligned} P(y \geq 3|N) &= 0.5 \\ &= 1 - \sum_{n=0}^{2} P(y = n|N) \\ &= 1 - \sum_{n=0}^{2} \binom{500}{n} P(x < 0.75)^{N-n} P(x >= 0.75)^n \end{aligned}$$

for N. Using numerical software, we find $N = 16806$ gives a probability of 50%.

(c) One exception to the assumptions mentioned above is that the returns are correlated and not independent. Another exception is that the stocks are not normally distributed. Depending on the correlations and distributions in the real world, the probabilities discussed above could change dramatically. For example, if the stocks had a bimodal distribution of winners and losers where winners had an annualized return of 312 % and the losers had an annualized return of 288 % (to keep the mean return at 12 %), the probability that a single stock exhibits an annualized return of at least 300 % could be near 50 %.

Chapter 13

Problem 13.1. Your research has identified a monthly signal with IR=1. You notice that delaying its implementation by one quarter reduces the IR to 0.75. What is the signal's half-life? What is the half-life of the value added?

Solution. We know that $\mathrm{IR}(t=0)=1$ and $\mathrm{IR}(t=3)=0.75$. We can also write $\mathrm{IR}(t=n)=\mathrm{IR}(t=0)\delta^n$ where n is the number of months we delay implementation. This implies that $\delta=(0.75)^{1/3}$. The half life is the number of months it takes for $\mathrm{IR}(t=n)=0.5\cdot\mathrm{IR}(t=0)$. Hence, we can write $1/2=\delta^\tau$ where τ is the half life. Solving for τ we find $\tau=3\ln(1/2)/\ln(0.75)=7.2$ months. Furthermore, we know that the value added is proportional to the square of the IR. Hence, we can write $1/2=\mathrm{VA}(t=0)/\mathrm{VA}(t=n)\propto(\mathrm{IR}(t=0)/\mathrm{IR}(t=n))^2=\delta^{2\tau}$. δ is the same as before, and so we find that the half life is $\tau=(3/2)(\ln(1/2)/\ln(3/4))$ which is half the half life of the information ratio, or about 3.6 months.

Problem 13.2. In further researching the signal in Problem 13.1, you discover that the correlation of active returns to this signal and this signal implemented 1 month late is 0.75. What is the optimal combination of current and lagged portfolios?

Solution. From equations 13.1 and 13.2, the optimal weight of the current portfolio is given by

$$\begin{aligned}
w^*_{Now} &= \frac{\delta+\frac{1-\delta}{1-\rho}}{\delta+1}\\
&= \frac{0.75^{1/3}+\frac{1-0.75^{1/3}}{1-0.75}}{0.75^{1/3}+1}\\
&= 0.67
\end{aligned}$$

The weight of the lagged portfolio $w^*_{Later}=1-w^*_{Now}$ is 0.33. Because $\rho<\delta$ we can combine the lagged portfolio with the current portfolio to diversify our holdings. In effect, this reinforces the signal while damping the noise.

Problem 13.3. You forecast $\alpha=2$ percent for a stock with $\omega=25$ percent, based on a signal with $\mathrm{IC}=0.05$. Suddenly the stock moves, with $\theta=10$ percent. How should you adjust your alpha? Is it now positive or negative?

Solution. We need to settle the old score. Using the forecasting rule of thumb, we can determine the old score as $s(-\Delta t)=\alpha/\mathrm{IC}\cdot\omega=0.02/(0.05\times0.25)=1.6$. Using equation 13.12, we can settle the old score according to

$$s^*(-\Delta t)=s(-\Delta t)-\frac{\mathrm{IC}\cdot r(-\Delta t,0)}{\sigma\sqrt{\Delta t}}$$

Assuming $\Delta t=1$ we have

$$\begin{aligned}
s^*(-\Delta t) &= 1.6-\frac{0.05\times0.1}{0.25}\\
&= 1.58
\end{aligned}$$

The revised α can then be calculated using the forecasting rule of thumb with the settled score as

$$\begin{aligned}
\alpha^*(-\Delta t) &= \omega\cdot\mathrm{IC}\cdot s^*(-\Delta t)\\
&= 0.25\times0.05\times1.58\\
&= 0.01975
\end{aligned}$$

So the revised α hardly changes from the original, which makes sense because θ is much less than a standard deviation away from the original prediction of α.

Chapter 14

Problem 14.1. Table 14.1 shows both alphas used in a constrained optimization and the modified alphas which, in an unconstrained optimization, lead to the same holdings. Comparing these two sets of alphas can help in estimating the loss in value added caused by the constraints. How? What is the loss in this example? The next chapter will discuss this in more detail.

Solution. We know that value added is proportional to the square of the information ratio (eq 5.12). We also know that the information ratio is proportional to the information coefficient (eq 6.1). As discussed in the text, the standard deviation of the constrained and unconstrained alphas, 0.57 and 2 percent respectively, imply a reduction in IC by 62 % for the constrained problem, according to reasoning from the forecasting rule of thumb. Since value added changes as the square of the IC, this implies a value loss of approximately 62% ($= 1 \times 1 - .62 \times .62$) due to the constraints.

Problem 14.2. Discuss how restrictions on short sales are both a barrier to a manager's effective use of information and a safeguard against poor information.

Solution. Restrictions on short sales act as a constraint on the manager. As shown in problem 14.1, constraints lead to an effective reduction in the information coefficient and hence, the manager cannot use his information in the most effective way possible. At the same time, if the manager has poor information, the restriction on short sales will limit the transactions he makes with this poor information, thus acting as a safeguard.

Problem 14.3. Lisa is a value manager who chooses stocks based on their price/earnings ratios. What biases would you expect to see in her alphas? How should she construct portfolios based on these alphas, in order to bet only on specific asset returns?

Solution. There could be industry and size biases in her alphas since PE ratios can depend on these factors. Lisa will want to make her alphas neutral to these factors before constructing her portfolio in order to bet only on the specific returns for each stock. One simple way to do this would be to calculate the capitalization weighted alpha for each industry and for companies of a given size, and then subtract these industry and size alphas from the PE alphas based on the exposure of the specific stock to these factors.

Problem 14.4. You are a benchmark timer who in backtests can add 50 basis points of risk-adjusted value added. You forecast 14 percent benchmark volatility, the recent average, but unfortunately benchmark volatility turns out to be 17 percent. How much value can you add, given this misestimation of benchmark volatility?

Solution. See figure 14.2. If the benchmark volatility turns out to be 17 percent, the percentage of value lost due to our forecast of 14 % is about 20%. This means that we will only be able to add about 40 basis points of risk-adjusted value. We can also calculate the loss according to equation 14.11 as

$$\begin{aligned}
\text{Loss} &= \text{VA}^* \cdot \left[1 - \left(\frac{\zeta}{\sigma}\right)^2\right]^2 \\
&= 50 \cdot \left[1 - \left(\frac{17}{14}\right)^2\right]^2 \\
&= 11.26
\end{aligned}$$

so that the value we can add is $50 - 11.26 = 38.74$ basis points, pretty close to our estimate based on the graph.

Problem 14.5. You manage 20 separate accounts using the same investment process. Each portfolio holds about 100 names, with 90 names appearing in all the accounts and 10 names unique to the particular account. Roughly how much dispersion should you expect to see?

Solution. We can calculate the expected dispersion according to equation 14.13

$$D = 2 \cdot \Phi^{-1}\left\{\left(\frac{1}{2}\right)^{1/N}\right\} \cdot \psi$$

Where $\Phi^{-1}(p)$ is the inverse normal CDF ($= \sqrt{2} \cdot \text{erf}^{-1}(2p-1)$), N is the total number of portfolios managed, and ψ is the average tracking error of each portfolio relative to the composite. From the problem, we know $N = 20$ and hence

$$\begin{aligned} D &= 2 \cdot \sqrt{2} \cdot \text{erf}^{-1}\left\{2\left(\frac{1}{2}\right)^{1/20} - 1\right\} \cdot \psi \\ &= 3.648 \cdot \psi \end{aligned}$$

Assuming, as in the text, that each stock has a risk of 20%, that the portfolios have equal weight of their constituent stocks and that the individual stocks are uncorrelated, leads to a tracking error for each portfolio of about $\sqrt{0.20}\%$ (since each portfolio has 10 out of 100 unique stocks and each unique stock contributes 0.2% to the total variance). This leads to a dispersion of 1.63% which agrees well with the results plotted in figure 14.3.

Chapter 15

Problem 15.1. Jill manages a long-only technology sector fund. Joe manages a risk-controlled, broadly diversified core equity fund. Both have information ratios of 0.5. Which would experience a larger boost in information ratio by implementing his or her strategy as a long/short portfolio? Under what circumstances would Jill come out ahead? What about Joe?

Solution. Long/short strategies offer the most upside when the universe of assets is large, the asset volatility is low and the strategy has high active risk. Since Jill manages a technology sector fund, this implies that her universe is limited. On the other hand, Joe's fund implies a large universe and low asset volatility. Hence, it seems as if Joe should experience a larger boost in his IR by implementing a long/short strategy. If Jill implemented a long/short strategy, she would come out ahead when the technology sector asset volatility was low and when her long/short portfolio had high active risk. If Joe implemented a long/short strategy, he would come out ahead when his portfolio has a high active risk.

Problem 15.2. You have a strategy with an information ratio of 0.5, following 250 stocks. You invest long-only, with active risk of 4 percent. Approximately what alpha should you expect? Convert this to the shrinkage in skill (measured by the information coefficient) induced by the long-only constraint.

Solution. From equation (15.11), we know that α can be approximated as

$$\begin{aligned}\alpha(\omega, N) &= 100 \cdot \text{IR} \cdot \left\{ \frac{\{1+\omega/100\}^{1-\gamma(N)} - 1}{1-\gamma(N)} \right\} \\ &= 100 \cdot 0.5 \cdot \left\{ \frac{\{1+4/100\}^{1-(53+250)^{0.57}} - 1}{1-(53+250)^{0.57}} \right\} \\ &= 1.25\%\end{aligned}$$

The shrinkage is given by equation (15.13) as

$$\begin{aligned}S &= \frac{\alpha(\omega, N)/\omega}{\text{IR}} \\ &= \frac{1.25\%}{4\% \times 0.5} \\ &= 0.625\end{aligned}$$

This is a substantial shrinkage factor!

Problem 15.3. How could you mitigate the negative size bias induced by the long-only constraint?

Solution. In this chapter we laired that the long-only constraint induces a negative size bias (a bias towards smaller companies). This bias could be mitigated by constraining the portfolio to have zero net exposure to size.