Stats 100C: Homework #4

Professor Arash Amini

Assignment: Bookwork 4.2, 4.6abc, 4.14abc, Additional: 4.1, 4.2, 4.3

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Additional Problems

Problem 4.1

(a) Suppose $n \geq 2$. Show that X has full column rank if and only if $s_{xx} \neq 0$.

(b) Show that the least squares estimate of $\boldsymbol{\beta}$ is given by

$$\hat{\beta}_1 = \frac{\bar{xy} - \bar{x}\bar{y}}{\bar{x^2} - (\bar{x})^2} =: \frac{s_{xy}}{s_{xx}}, \quad \hat{\beta}_0 = \bar{y} - \beta_1 \bar{x} \tag{1}$$

(c) Show the alternative expressions for s_{xy} and s_{xx},

$$s_{xy} = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}), \quad s_{xx} = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 \tag{2}$$

Solution

(a)

($\Rightarrow$) Suppose X is full column rank. Suppose for contradiction that $s_{xx} = 0$. Then,

$$\begin{aligned}
\sum_{i=1}^{n}(x_i - \bar{x})^2 = 0 &\Leftrightarrow x_{i1} - \bar{x} = 0, \forall i \\
&\Leftrightarrow x_{i1} = \bar{x}, \forall i \\
&\Leftrightarrow \exists k \in \mathbb{R} : k = \bar{x} \\
&\Rightarrow k \cdot \mathbf{1} = \mathbf{x}_1
\end{aligned}$$

This implies that $\mathbf{x}_1$ is a scalar multiple of the first column of X, which contradicts the assumption that X is full column rank. Thus, we conclude that $s_{xx} = 0$.

($\Leftarrow$) Suppose $s_{xx} \neq 0$. Now suppose for contradiction that X is not full column rank. Then the columns of X are linearly dependent, i.e.,

$$\begin{aligned}
\exists k \in \mathbb{R} : \mathbf{x}_1 = k \cdot \mathbf{1} &\Leftrightarrow \mathbf{x}_{i1} = k, \forall i \\
&\Leftrightarrow \bar{x} = c \\
&\Leftrightarrow s_{xx} = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 = 0
\end{aligned}$$

which contradicts are assumption that $x_{xx} \neq 0$. We conclude that X must be full rank. Both directions hold. $\square$