Stats 100C: Homework #3

Professor Arash Amini

Assignment: 3.1, 3.2, 3.10, 3.11, 3.14, 4.5

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Problem 3.1

Let $= (x_1, x_2, \ldots, x_n) \sim N(\mathbf{0}, I_n)$ be a MVN random vector in $\mathbb{R}^n$.

(a) Let $U \in \mathbb{R}^{n \times n}$ be an orthogonal matrix, and find the distribution of $U'\mathbf{x}$.
Since $U'\mathbf{x}$ is a linear transformation of $\mathbf{x}$, $U'\mathbf{x}$ is also MVN.

$$\begin{aligned}
&\mathrm{E}\left(U'\mathbf{x}\right) = U'\mathrm{E}(\mathbf{x}) = U'\mathbf{0} = \mathbf{0} \\
&\mathrm{Var}(U'\mathbf{x}) = U'\mathrm{Var}('\mathbf{x})U = U'IU = I_n \\
&\Rightarrow U'\mathbf{x} \sim N(\mathbf{0}, I_n)
\end{aligned}$$

Let $\mathbf{y} \sim N(\mathbf{0}, \Sigma)$ be a MVN random vector in $\mathbb{R}^n$. Let $\Sigma = U\Lambda U'$ be the spectral decomposition of Σ.

(b) Is the claim that the diagonal elements of Λ are nonnegative true?
Σ is a covariance matrix, so Σ is positive semi-definite, which implies that its eigenvalues are nonnegative. Since Λ is a diagonal matrix with the eigenvalues of Σ on the diagonal, then we conclude that the diagonal elements of Λ are nonnegative.

(c) Let $\mathbf{z} = U'\mathbf{y}$. Find the distribution of $\mathbf{z}$.
Since $\mathbf{z}$ is a linear transformation of a MVN vector, $\mathbf{z}$ is also a MVN vector.

$$\begin{aligned}
&\mathrm{E}(\mathbf{z}) = U'\mathrm{E}(\mathbf{y}) = U'\mathbf{0} = \mathbf{0} \\
&\mathrm{Var}(\mathbf{z}) = U'\mathrm{Var}(\mathbf{y})U = U'\Sigma U = U'(U\Lambda U')U = \Lambda \\
&\Rightarrow \mathbf{z} \sim N(\mathbf{0}, \Lambda)
\end{aligned}$$

(d) Since $\mathbf{z}$ is MVN, then uncorrelatedness is equivalent to independence. Since $\mathrm{Cov}(\mathbf{z}) = \Lambda$, which is a diagonal matrix, then $\mathrm{Cov}(z_i, z_j) = 0, \forall i \neq j$, and we conclude that the components of $\mathbf{z}$ are independent.

(e) Since Λ is diagonal, $\mathrm{Cov}(z_i, z_j) = 0, \forall i \neq j$. Thus, $\mathrm{Var}(z_i) = \Lambda_{ii}$, the i-th element on the diagonal.

(f) Let $\mathbf{a} = (a_1, \ldots a_n) \in \mathbb{R}^n$ be a fixed, non-random vector. Find the distribution of $\mathbf{a}'\mathbf{z}$.
Since $\mathbf{a}'\mathbf{z}$ is a linear transformation of $\mathbf{z}$, which is MVN, then $\mathbf{a}'\mathbf{z}$ is also MVN.

$$\begin{aligned}
&\mathrm{E}(\mathbf{a}'\mathbf{z}) = \mathbf{a}'\mathrm{E}(\mathbf{z}) = \mathbf{a}'\mathbf{0} = 0 \\
&\mathrm{Var}(\mathbf{a}'\mathbf{z}) = \mathbf{a}'\mathrm{Var}(\mathbf{z})\mathbf{a} = \mathbf{a}'\Lambda\mathbf{a} \\
&\Rightarrow \mathbf{a}'\mathbf{z} \sim N(0, \mathbf{a}'\Lambda\mathbf{a})
\end{aligned}$$

(g) Suppose $\Lambda_{ii} > 0, \forall i$. Specify an $\mathbf{a}$ such that $\text{Var}(\mathbf{a}'\mathbf{z}) = 1$.

$$\text{Var}(\mathbf{a}'\mathbf{z}) = 1 \Leftrightarrow \mathbf{a}\Lambda\mathbf{a} = 1 \Leftrightarrow \sum_{i=1}^{n} a_i^2 \Lambda_{ii}^2 = 1, \quad a_i = \frac{1}{\sqrt{n}\sqrt{\Lambda_{ii}}}$$

Thus, one choice for the vector $\mathbf{a}$ is:

$$\frac{1}{\sqrt{n}} \begin{bmatrix} \frac{1}{\sqrt{\Lambda_{11}}} \\ \frac{1}{\sqrt{\Lambda_{22}}} \\ \vdots \\ \frac{1}{\sqrt{\Lambda_{nn}}} \end{bmatrix}$$

(h) Let $\mathbf{u}_1 \in \mathbb{R}^n$ be the first column of U. Find the joint distribution of $(\mathbf{a}'\mathbf{z}, \mathbf{u}_1'\mathbf{y}) \in \mathbb{R}^2$.

Note that $\mathbf{z} = U'\mathbf{y} \Rightarrow \mathbf{y} = U\mathbf{z}$, since U is orthogonal. Then we can write the vector $(\mathbf{a}'\mathbf{z}, \mathbf{u}_1'\mathbf{y})$ as

$$\begin{bmatrix} \mathbf{a}'\mathbf{z} \\ \mathbf{u}_1'\mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{a}'\mathbf{z} \\ \mathbf{u}_1'U\mathbf{z} \end{bmatrix} = \begin{bmatrix} \mathbf{a}'\mathbf{z} \\ (1, 0, \dots, 0)\mathbf{z} \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & \dots & a_n \\ 1 & 0 & \dots & 0 \end{bmatrix} \mathbf{y}$$

The second equality holds because U is orthogonal. Since the transformations are linear, $(\mathbf{a}'\mathbf{z}, \mathbf{u}_1'\mathbf{y})$ is MVN.

$$\text{E}\left(\begin{bmatrix} \mathbf{a}'\mathbf{z} \\ \mathbf{u}_1'\mathbf{y} \end{bmatrix} \right) = \begin{bmatrix} \mathbf{a}'\text{E}(\mathbf{z}) \\ \mathbf{u}_1'\text{E}(\mathbf{y}) \end{bmatrix} = \mathbf{0}$$

$$\text{Var}\left(\begin{bmatrix} \mathbf{a}'\mathbf{z} \\ \mathbf{u}_1'\mathbf{y} \end{bmatrix} \right) = \begin{bmatrix} a_1 & a_2 & \dots & a_n \\ 1 & 0 & \dots & 0 \end{bmatrix} \text{Var}(\mathbf{z}) \begin{bmatrix} a_1 & 1 \\ a_2 & 0 \\ \vdots & \vdots \\ a_n & 0 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} a_i^2 \Lambda_{ii} & a_1\Lambda_{11} \\ a_1\Lambda_{11} & \Lambda_{11} \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{a}'\mathbf{z} \\ \mathbf{u}_1'\mathbf{y} \end{bmatrix} \sim N\left(\mathbf{0}, \begin{bmatrix} \sum_{i=1}^{n} a_i^2 \Lambda_{ii} & a_1\Lambda_{11} \\ a_1\Lambda_{11} & \Lambda_{11} \end{bmatrix} \right)$$

□

Problem 3.2
Let $H \in \mathbb{R}^{n\times n}$ be symmetric and idempotent, hence a projection matrix. Let $\mathbf{x} \sim N(0, I_n)$.
Solution
(a) Let $\sigma > 0$ be a positive number. Find the distribution of σx.
Since σx is a scalar multiple of a MVN random vector, then σx is also MVN.

$$\begin{aligned}
&\mathrm{E}(\sigma\mathbf{x}) = \sigma\mathrm{E}(\mathbf{x}) = \mathbf{0}\\
&\mathrm{Var}(\sigma\mathbf{x}) = \sigma^2\mathrm{Var}(\mathbf{x}) = \sigma^2 I_n\\
&\Rightarrow \sigma\mathbf{x} \sim N(\mathbf{0}, \sigma^2 I_n)
\end{aligned}$$

(b) Let $\mathbf{u} = H\mathbf{x}, \mathbf{v} = (I - H)\mathbf{x}$. Find the joint distribution $(\mathbf{u}, \mathbf{v})'$
First note that since H is symmetric idempotent, we have the following three results:

$$\begin{aligned}
&H(I-H)' = H - HH' = H - HH = H - H = 0\\
&(I-H)H' = H' - HH' = H - HH = H - H = 0\\
&(I-H)(I-H)' = I - H - H' + HH' = I - 2H + H = I - H
\end{aligned}$$

We can write the vector $(\mathbf{u}, \mathbf{v})'$ as a product of a block matrix and $\mathbf{x}$, and since $(\mathbf{u}, \mathbf{v})'$ can be written as a linear transformation a MVN random vector, we know that $(\mathbf{u}, \mathbf{v})'$ is also MVN.

$$\begin{bmatrix}\mathbf{u}\\ \mathbf{v}\end{bmatrix} = \left[\begin{array}{c} H \\ \hline (I-H)\end{array}\right]\mathbf{x}, \qquad \mathrm{E}\left(\begin{bmatrix}\mathbf{u}\\ \mathbf{v}\end{bmatrix}\right) = \left[\begin{array}{c} H \\ \hline (I-H)\end{array}\right]\mathrm{E}(\mathbf{x}) = \mathbf{0}$$

$$\mathrm{Cov}\left(\begin{bmatrix}\mathbf{u}\\ \mathbf{v}\end{bmatrix}\right) = \left[\begin{array}{c} H \\ \hline (I-H)\end{array}\right] I_n \left[\begin{array}{c|c} H' & (I-H)'\end{array}\right] = \left[\begin{array}{c|c} H & H(I-H)' \\ \hline (I-H)H' & (I-H)(I-H)'\end{array}\right] = \left[\begin{array}{c|c} H & 0 \\ \hline 0 & (I-H)\end{array}\right]$$

$$\begin{bmatrix}\mathbf{u}\\ \mathbf{v}\end{bmatrix} \sim N\left(\mathbf{0}, \left[\begin{array}{c|c} H & 0 \\ \hline 0 & (I-H)\end{array}\right]\right)$$

(c) Is the claim that $\mathbf{u}$ and $\mathbf{v}$ independent true?
Since $(\mathbf{u}, \mathbf{v})$ is a MVN random vector, uncorrelatedness is equivalent to independence. The distribution found in part (b) that the covariance matrix is block-diagonal. Thus, $\mathbf{u}$ and $\mathbf{v}$ are uncorrelated, and thus independent.

(d) Let $\boldsymbol{\mu} \in \mathrm{Im}(H)$. Show that $H\boldsymbol{\mu} = \boldsymbol{\mu}$.
Since $\boldsymbol{\mu} \in \mathrm{Im}(H)$, there exists some $\mathbf{y} \in \mathbb{R}^n$ such that $\boldsymbol{\mu} = H\mathbf{y}$. Then we can left multiply by H to get: $H\boldsymbol{\mu} = H(H\mathbf{y}) = H\mathbf{y} = \boldsymbol{\mu}$, since H is idempotent.

(e) Assume that $\mathbf{1} \in \mathrm{Im}(H)$. Find the distribution of $\mathbf{1}'H\mathbf{x}$.
Since $\mathbf{1}'H\mathbf{x}$ is a linear transformation of a MVN random vector, it is also normally distributed.

$$\begin{aligned}
&\mathrm{E}(\mathbf{1}'H\mathbf{x}) = \mathbf{1}'H\mathrm{E}(\mathbf{x}) = 0\\
&\mathrm{Var}(\mathbf{1}'H\mathbf{x}) = \mathbf{1}'H\mathrm{Var}(\mathbf{x})(\mathbf{1}'H)' = \mathbf{1}'HH'\mathbf{1} = \mathbf{1}'H\mathbf{1}
\end{aligned}$$

By part (d), we know that since $\mathbf{1} \in \mathrm{Im}(H), H\mathbf{1} = \mathbf{1}$, so $\mathrm{Var}(\mathbf{1}'H\mathbf{x}) = \mathbf{1}'\mathbf{1} = n$, and we conclude that

$$\mathbf{1}'h\mathbf{x} \sim N(0, n)$$

□

Problem 3.10
The regression model for the hardness data is given as:

$$y = \beta_0 + \beta_1 x_1 + \epsilon$$

The 14×2 matrix X and the 14×1 vector of responses $\mathbf{y}$ are given by

$$X = \begin{bmatrix} 1 & 30 \\ 1 & 30 \\ 1 & 30 \\ 1 & 30 \\ 1 & 40 \\ 1 & 40 \\ 1 & 40 \\ 1 & 50 \\ 1 & 50 \\ 1 & 50 \\ 1 & 60 \\ 1 & 60 \\ 1 & 60 \\ 1 & 60 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} 55.8 \\ 59.1 \\ 54.8 \\ 54.6 \\ 43.1 \\ 42.2 \\ 45.2 \\ 31.6 \\ 30.9 \\ 30.8 \\ 17.5 \\ 20.5 \\ 17.2 \\ 16.9 \end{bmatrix}, \quad X'X = \begin{bmatrix} 14 & 630 \\ 630 & 30300 \end{bmatrix}, \quad (X'X)^{-1} = \begin{bmatrix} 1.1099 & -0.0231 \\ -0.0231 & 0.0005 \end{bmatrix}$$

Then the expression for the least squares estimates in $\hat{\boldsymbol{\beta}} = (X'X)^{-1} X'\mathbf{y}$ is given by

$$\hat{\boldsymbol{\beta}} = \begin{bmatrix} 1.1099 & -0.0231 \\ -0.0231 & 0.0005 \end{bmatrix} X'\mathbf{y} = \begin{bmatrix} 94.13 \\ -1.27 \end{bmatrix}$$

□

Problem 3.11
Consider a trivariate normal distribution

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, \quad \mathrm{E}[\mathbf{y}] = \boldsymbol{\mu}_y = \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix}, \quad \mathrm{Cov}(\mathbf{y}) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

(a) Determine the marginal bivariate distribution of $(y_1, y_2)'$.
(b) Determine the conditional bivariate distribution of $(y_1, y_2)'$, given that $y_3 = 5$.

Solution
(a) We write $(y_1, y_2)'$ as a linear transformation of $\mathbf{y}$ and then take expectation and variance,

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \mathbf{y}$$

$$\mathrm{E}\left(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \boldsymbol{\mu}_y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

$$\mathrm{Cov}\left(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \mathrm{Cov}(\mathbf{y}) \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \sim N\left(\begin{bmatrix} 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \right)$$

(b) First note that the $\mathrm{Cov}(\mathbf{y})$ can be written as the following block matrix

$$\mathrm{Cov}(\mathbf{y}) = \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 2 & -1 \\ \hline 1 & -1 & 3 \end{array} \right] = \begin{bmatrix} \Sigma_{\star\star} & \Sigma_{\star 3} \\ \Sigma_{3\star} & \Sigma 33 \end{bmatrix}$$

The conditional bivariate distribution of $\mathbf{y}^\star = (y_1, y_2)'$ given $y_3 = 5$ is also MVN, and it can be calculated as follows:

$$\boldsymbol{\mu}_{\star|3} = \boldsymbol{\mu}_\star + \Sigma_{\star 3} \Sigma 33^{-1} (\mathbf{y}_3 \boldsymbol{\mu}_3) = \begin{bmatrix} 2 \\ 6 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cdot \frac{1}{3}(5-4) = \begin{bmatrix} 7/3 \\ 17/3 \end{bmatrix}$$

$$\Sigma_{\star|3} = \Sigma_{\star\star} - \Sigma_{\star 3} \Sigma_{33}^{-1} \Sigma_{\star 3}' = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cdot \frac{1}{3} \cdot \begin{bmatrix} 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 1/3 & -1/3 \\ -1/3 & 1/3 \end{bmatrix} = \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 5/3 \end{bmatrix}$$

$$\mathbf{y}^\star | \mathbf{y}_3 = 5 \sim N\left(\begin{bmatrix} 7/3 \\ 17/3 \end{bmatrix}, \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 5/3 \end{bmatrix} \right)$$

□

Problem 3.14

Suppose that the covariance matrix of a vector $\mathbf{y}$ is $\sigma^2 I_n$. Find the covariance matrix of

(a) $A\mathbf{y}$

(b) $H\mathbf{y}$

(c) $(I-H)\mathbf{y}$

(d) $\begin{bmatrix} A \\ I-H \end{bmatrix}\mathbf{y}$

$$H = X\left(X'X\right)^{-1}X', \quad A = \left(X'X\right)^{-1}X'$$

Solution

Recall that $H, (I-H)$ are symmetric and idempotent.

(a) $\text{Cov}(A\mathbf{y}) = A\text{Cov}(\mathbf{y})A' = A\sigma^2 IA' = \sigma^2 AA' = \sigma^2(X'X)^{-1}X'X(X'X)^{-1} = \sigma^2(X'X)^{-1}$

(b) $\text{Cov}(H\mathbf{y}) = H\sigma^2 IH' = \sigma^2 HH' = \sigma^2 HH = \sigma^2 H$

(c) $\text{Cov}((I-H)\mathbf{y}) = (I-H)\sigma^2 I(I-H)' = \sigma^2(I-H)(I-H)' = \sigma^2(I-H)(I-H) = \sigma^2(I-H)$

(d) Recall from the Problem 3.13 (Homework 1) that $A(I-H) = (I-H)A' = 0$.

$$\begin{aligned}
\text{Cov}\left(\begin{bmatrix} A \\ I-H \end{bmatrix}\mathbf{y}\right) = \begin{bmatrix} A \\ I-H \end{bmatrix}\text{Cov}\left(\mathbf{y}\right)\begin{bmatrix} A' & (I-H)' \end{bmatrix} &= \sigma^2\begin{bmatrix} A \\ I-H \end{bmatrix}\begin{bmatrix} A' & (I-H)' \end{bmatrix} \\
&= \sigma^2\begin{bmatrix} A \\ I-H \end{bmatrix}\begin{bmatrix} A' & (I-H) \end{bmatrix} \\
&= \sigma^2\begin{bmatrix} (X'X)^{-1} & A(I-H) \\ (I-H)A' & (I-H)(1-H)' \end{bmatrix}
\end{aligned}$$

$$\text{Cov}\left(\begin{bmatrix} A \\ I-H \end{bmatrix}\mathbf{y}\right) = \sigma^2\left[\begin{array}{c|c} (X'X)^{-1} & 0 \\ \hline 0 & (I-H) \end{array}\right]$$

□

Problem 4.5 After fitting the regression model,

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \epsilon \tag{1}$$

on 15 cases, it is found that the mean squared error $s^2 = 3$, and

$$(X'X)^{-1} \begin{bmatrix} 0.5 & 0.3 & 0.2 & 0.6 \\ 0.3 & 6.0 & 0.5 & 0.4 \\ 0.2 & 0.5 & 0.2 & 0.7 \\ 0.6 & 0.4 & 0.7 & 3.0 \end{bmatrix}$$

Find:
(a) The estimate of $\text{Var}(\hat{\beta}_1)$.
(b) The estimate of $\text{Cov}(\hat{\beta}_1, \hat{\beta}_3)$
(c) The estimate of $\text{Cor}(\hat{\beta}_1, \hat{\beta}_3)$
(d) The estimate of $\text{Var}(\hat{\beta}_1 - \hat{\beta}_3)$

Solution
(a) We know that $\hat{\boldsymbol{\beta}} \sim N\left(\boldsymbol{\beta}, \sigma^2 (X'X)^{-1}\right)$, and since s^2 is an unbiased estimator for σ^2, then we can calculate $\text{Var}(\hat{\boldsymbol{\beta}})$,

$$\text{Var}\left(\boldsymbol{\beta}\right) = s^2 \left(X'X\right)^{-1} = 3 \cdot \begin{bmatrix} 0.5 & 0.3 & 0.2 & 0.6 \\ 0.3 & 6.0 & 0.5 & 0.4 \\ 0.2 & 0.5 & 0.2 & 0.7 \\ 0.6 & 0.4 & 0.7 & 3.0 \end{bmatrix} = \begin{bmatrix} 1.50 & 0.90 & 0.60 & 1.80 \\ 0.90 & 18.0 & 1.50 & 1.20 \\ 0.60 & 1.50 & 0.60 & 2.10 \\ 1.80 & 1.20 & 2.10 & 9.00 \end{bmatrix}$$

Then $\text{Var}(\hat{\beta}_1)$ can be read off of the covariance matrix calculated above. $\text{Var}(\hat{\beta}_1) = 18.0$.

(b) $\text{Cov}(\hat{\beta}_1, \hat{\beta}_3) = \left[\text{Cov}(\hat{\boldsymbol{\beta}})\right]_{13} = 1.20$

(c) $\text{Cor}(\hat{\beta}_1, \hat{\beta}_3) = \frac{\text{Cov}(\hat{\beta}_1, \hat{\beta}_3)}{\sqrt{\text{Var}(\hat{\beta}_1)}\sqrt{\text{Var}(\hat{\beta}_3)}} = \frac{1.20}{\sqrt{18 \cdot 9}} = 0.094$

(d) $\text{Var}(\hat{\beta}_1 - \hat{\beta}_3) = \text{Var}(\hat{\beta}_1) + \text{Var}(\hat{\beta}_3) - 2\text{Cov}(\hat{\beta}_1, \hat{\beta}_3) = 18.0 + 9.0 - 2 \cdot 1.20 = 24.6$