STATS 200A: Homework #7

Professor Yingnian Wu
Assignment: 1, 2

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Problem 1

Prove the L2 Strong Law of Large Numbers:

Let $X_1, X_2, \ldots, X_n$ be independent random variables, where $\mathrm{E}(X_i) = \mu$ for all i and $\mathrm{Var}(X_i) \leq B < \infty$. Define $\overline{X_n} := \frac{1}{n}\sum_{i=1}^{n} X_i$. Then

$$\Pr\left(\overline{X_n} \to \mu\right) = 1 \tag{1}$$

Solution
Note that in the strong law of large numbers, though the sequence of random variables is infinitely long, we consider the first n elements of these sequences. We can assume without loss of generality that $\mu = 0$, else we can consider the random variable obtained when subtracting the mean from each X_i. Let $\varepsilon > 0$ and define $S_n := \sum_{i=1}^{n} X_i$. Then to prove the equality in (1) above, it suffices to show that $\Pr(|S_n - 0| > n\varepsilon) \to 0$. We do this by providing a bound for $|S_k/k|$.

Using Chebyshev's Inequality, we can write

$$\begin{aligned}
\Pr(|S_n| > n\varepsilon) &\leq \frac{\mathrm{E}(|S_n|^2)}{(n\varepsilon)^2} && (2)\\
&\leq \frac{\mathrm{E}(X_1^2) + \ldots + \mathrm{E}(X_n^2)}{n^2\varepsilon^2} && (3)\\
&\leq \frac{nB}{n^2\varepsilon^2} && (4)\\
&= \frac{B}{n\varepsilon^2} && (5)
\end{aligned}$$

where the inequality in 3 holds because $\mathrm{E}(X_i) = \mu = 0$ for all i, so terms of the form $\mathrm{E}(X_i)\mathrm{E}(X_j) = 0$, leaving only the $\mathrm{E}(X_i^2)$ terms. These are nothing but the variance terms, which are assumed to be uniformly bounded by $B < \infty$. Then, if we consider the subsequence S_{n^2}, we can use the Borel-Cantelli Lemma to conclude that

$$\Pr(|S_{n^2}| > n^2\varepsilon \textit{ infinitely often}) = 0 \Rightarrow \frac{S_{n^2}}{n^2} \to 0 \tag{6}$$

If we further define $D_n = \max_{n^2 \leq k < (n+1)^2} |S_k - S_{n^2}|$ and calculate the expectation,

$$\begin{aligned}
\mathrm{E}(D_n^2) &= \mathrm{E}\left(\max_{n^2 \leq k < (n+1)^2} \left|S_k - S_{n^2}^2\right|\right) && (7)\\
&\leq \mathrm{E}\left(\sum_{n^2 \leq k \leq (n+1)^2} \left|S_k - S_{n^2}^2\right|\right) && (8)\\
&= \sum_{n^2 \leq k \leq (n+1)^2} \mathrm{E}(|S_k - S_n|^2) && (9)
\end{aligned}$$

Since each term in the sum has $k - n^2$ terms, and each individual term $\mathrm{E}[X_i^2] \leq B$, then we can rewrite the last inequality as

$$\begin{aligned}
\sum_{n^2 \leq k \leq (n+1)^2} \mathrm{E}(|S_k - S_n|^2) &\leq \sum_{n^2 \leq k \leq (n+1)^2} (k - n^2)B && (10)\\
&\leq 4n^2 B && (11)
\end{aligned}$$

and again, by Borel-Cantelli, we conclude that $\frac{D_n}{n^2} \to 0$ almost surely. At this point, we have the sufficient bounds to complete the proof. For any k, we can then use the following bound,

$$\left|\frac{S_k}{k}\right| \leq \frac{|S_{n^2}| + D_n}{n^2} \tag{12}$$

where $n = \lfloor \sqrt{k} \rfloor$. From above, we've shown that $S_{n^2}/n \to 0$ almost surely and $D_n/n^2 \to 0$ almost surely, so $|S_k/k| \to 0$, and $\Pr(S_n > n\varepsilon) \to 0$ and the equality in (1) holds. $\square$

Problem 2

Prove the Lindeberg Central Limit Theorem by assuming the existence of $\mathrm{E}(|X|^3)$

Solution
Let $X_1, X_2, \ldots, X_n$ be independent random variables with $\mathrm{E}(X_i) = 0$ for all i and variances $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$, with

$$\tau_n^2 := \sigma_1^2 + \ldots + \sigma_n^2 > 0 \tag{13}$$

for all n. In addition, we denote $S_n := X_1 + \ldots + X_n$. Suppose, in addition, that the Lindeberg condition is satisfied and that $\mathrm{E}(|X_i|^3)$ exists. That is, for any $\epsilon > 0$,

$$\frac{1}{\tau_n^2} \sum_{i=1}^{n} \mathrm{E}(X_i^2 \cdot \mathbf{1}_{\{|X_i| > \epsilon \tau_n\}}) \to 0 \tag{14}$$

as $n \to \infty$. With these assumptions, we will prove that for any $a \in \mathbb{R}$,

$$\Pr(S_n/\tau_n \leq a) - \Phi(a) \to 0 \tag{15}$$

as $n \to \infty$. In other words, S_n/τ_n converges in distribution to the standard normal distribution. If we consider functions that are 3-times differentiable, then we can use the Berry-Esseen theorem to find the following bound

$$\mathrm{E}\left[|f(S_n/\tau_n|\right) - \mathrm{E}(f(Z)] \leq \mathrm{E}[|X|^3] \cdot \frac{1}{\sqrt{n}} \cdot \left| f^{(3)}(S_n) \right| \to 0 \tag{16}$$

as $n \to \infty$, where $Z \sim N(0,1)$. Consider

$$f_a(x) = I_{(-\infty,a]} = \begin{cases} 1 & \text{if } x \leq a \\ 0 & \text{for } x > a \end{cases}$$

Then, if we can show that this is bounded above and below by two smooth functions, then we will be able squeeze the difference in expectation of this step function between the difference obtained from the Berry-Esseen theorem in (16), and the Central Limit Theorem would follow. For any $\delta > 0$, if we consider the function f defined on $[a, a+\delta]$, then we see that f is monotonically decreasing on this interval. From this, we obtain the following inequality,

$$f_a(x) \leq f(x) \leq f_{a+\delta}(x) \tag{17}$$

for all $x \in \mathbb{R}$. Note that if we shift by δ, then

$$f_{a+\delta}(x+\delta) = \begin{cases} 1 & \text{if } x + \delta \leq a + \delta \Leftrightarrow x \leq a \\ 0 & \text{for } x + \delta > a + \delta \Leftrightarrow x > \delta \end{cases}$$

so $f_{x+\delta}(x+\delta) = f_a(x)$. Using this with the inequality given in (17), we have

$$f_a(x) = f_{a+\delta}(x+\delta) \geq f(x+\delta) \tag{18}$$
$$\Rightarrow f(x+\delta) \leq f_a(x) \leq f(x) \quad \forall x \in \mathbb{R} \tag{19}$$

giving us both an upper and lower found for the step function f_a. If we take $x := \frac{S_n}{\tau_n}$, then we have

$$f\left(\frac{S_n}{\tau_n} + \delta\right) \leq f_a\left(\frac{S_n}{\tau_n}\right) \leq f\left(\frac{S_n}{\tau_n}\right) \tag{20}$$
$$\Rightarrow \mathrm{E}\left[f\left(\frac{S_n}{\tau_n} + \delta\right)\right] \leq \mathrm{E}\left[f_a\left(\frac{S_n}{\tau_n}\right)\right] \leq \mathrm{E}\left[f\left(\frac{S_n}{\tau_n}\right)\right] \tag{21}$$

Since we took f to be a smooth function, we can apply our results from (16) to conclude that

$$\mathrm{E}\left[f\left(Z+\delta\right)\right] \leq \mathrm{E}\left[f_a\left(Z\right)\right] \leq \mathrm{E}\left[f\left(f(Z)\right)\right] \quad \forall \delta > 0 \tag{22}$$

Thus, if we fix $\delta > 0$ and define such a function f, for $a-\delta \leq Z \leq a+\delta$, we have

$$\begin{aligned} \left|\mathrm{E}\left[f\left(Z+\delta\right)\right] - \mathrm{E}\left[f\left(Z\right)\right]\right| &= \left|\mathrm{E}\left[f\left(\right) - f(Z)\right] \cdot \mathbf{1}_{\{a-\delta\leq Z\leq a+\delta\}}\right| \leq \left|\mathrm{E}\left[\mathbf{1}_{\{a-\delta\leq Z\leq a+\delta\}}\right]\right| && (23)\\ &= \Pr(a-\delta \leq Z \leq a+\delta) && (24)\\ &= \Phi(a+\delta) - \Phi(a-\delta) && (25)\\ &\leq \Phi(\delta) - \Phi(-\delta) && (26)\\ &\leq 2\delta\Phi(0) && (27)\\ &= \frac{2\delta}{\sqrt{2\pi}} && (28) \end{aligned}$$

Taking $\delta := \epsilon\sqrt{2\pi}/6$ and letting $n \to \infty$, then we conclude

$$\left|\mathrm{E}\left[f_n\left(\frac{S_n}{\tau_n}\right)\right] - \mathrm{E}\left[f_n(Z)\right]\right| \leq \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon \tag{29}$$

The Central Limit Theorem follows. □