STATS 200A: Homework #3

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Assignment: 1, 2

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Problem 1

For $X \sim \text{Binom}(n, p)$, let $\mu = np$ and $\sigma^2 = np(1-p)$. Let $Z = (X - \mu)/\sigma$. Prove that $P(Z \in [a, b]) \to \int_a^b f(z)dz$, where $f(z)$ is the density of the standard normal distribution.

Solution

For general $p \in (0, 1)$, let $q = 1 - p$. Let $k = \mu + z\sigma = np + z\sqrt{npq} = np + d$. Neglecting $O(1/n)$ terms, we have

$$
\begin{aligned}
P(X = np + d) &= \frac{n!}{(np+d)!(nq-d)!} p^{np+d} q^{nq-d} &(1)\\
&= \frac{\sqrt{2} n^n e^{-n} p^{np+d} q^{nq-d}}{\sqrt{2\pi(np+d)}(np+d)^{np+d} e^{-np-d} \sqrt{2\pi(nq-d)}(nq-d)^{nq-d} e^{-nq+d}} &(2)\\
&= \frac{\sqrt{n}}{\sqrt{2\pi(np+d)(nq-d)}} \left(\frac{np}{np+d}\right)^{np+d} \left(\frac{nq}{nq-d}\right)^{nq-d} &(3)
\end{aligned}
$$

Taking the log of both sides and using the Taylor expansion of $\log(1+\delta) = \delta - \delta^2/2 + O(\delta^3)$, we have

$$
\begin{aligned}
&\log\left[\left(\frac{np}{np+d}\right)^{np+d} \left(\frac{nq}{nq-d}\right)^{nq-d}\right] &(4)\\
&= \ -(np+d)\log\left(1 + \frac{d}{np}\right) - (nq-d)\log\left(1 - \frac{d}{nq}\right) &(5)\\
&= (-np+d)\left(\frac{d}{np} - \frac{d^2}{2n^2p^2} + O\left(\frac{1}{n^3}\right)\right) - (nq-d)\left(-\frac{d}{nq} - \frac{d^2}{2n^2q^2} + O\left(\frac{1}{n^3}\right)\right) &(6)\\
&= -d + \frac{d^2}{2np} - \frac{d^2}{2np} + O\left(\frac{1}{n^3}\right) + d + \frac{d^2}{2nq} - \frac{d^2}{nq} + O\left(\frac{1}{n^3}\right) &(7)\\
&= -\frac{d^2}{2np} - \frac{d^2}{2nq} + O\left(\frac{1}{n^3}\right) &(8)\\
&= -z^2\left(\frac{npq}{2np} + \frac{npq}{nq}\right) &(9)\\
&= -z^2\left(\frac{q}{2} + \frac{p}{2}\right) &(10)\\
&= \frac{-z^2}{2} &(11)
\end{aligned}
$$

Note also that in equation (3), since $k = np + d$ and $n - k = n - np - d = nq - d$, we can then express the first term of the product as

$$\frac{\sqrt{n}}{\sqrt{2\pi(np+d)(nq-d)}} = \sqrt{\frac{1}{2\pi \frac{1}{n} k(n-k)}} = \frac{1}{\sqrt{2\pi \frac{k}{n} n(1-\frac{k}{n})}} \tag{12}$$

Since $k = np + d$, then as $n \to \infty$, we have $\frac{k}{n} \to p$, so equation (12) gives us

$$\frac{1}{\sqrt{2\pi \frac{k}{n} n(1-\frac{k}{n})}} \to \frac{1}{\sqrt{2\pi pnq}} \tag{13}$$

Combining our conclusions from equations (3), (11), and (13), we have the following

$$P(X = np + d) = \frac{1}{\sqrt{2\pi npq}} e^{-z^2/2} \tag{14}$$

Taking $\Delta z = \sqrt{npq}$, equation (14) becomes

$$P(X = np + d) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \Delta z \tag{15}$$

$$= f(z) \cdot \Delta z \tag{16}$$

Since we are calculating the probability of $Z \in [a, b]$, we consider the following quantities

$$Z = a = \frac{X - \mu}{\sigma}$$
$$\Rightarrow X = a\sigma + \mu$$
$$Z = b = \frac{X - \mu}{\sigma}$$
$$\Rightarrow X = b\sigma + \mu$$

Then we can calculate the probability that the point falls within this region

$$P(Z \in [a, b]) = P(X \in [a\sigma + \mu, b\sigma + \mu]) \tag{17}$$

$$= \sum_{k=\mu+a\sigma}^{\mu+b\sigma} P(X = k) \tag{18}$$

$$= \sum_{z \in [a,b]} P(X = \mu + z\sigma) \tag{19}$$

$$= \sum_{z \in [a,b]} f(z) \cdot \Delta z \tag{20}$$

As $n \to \infty$, $\Delta z = \frac{1}{\sqrt{npq}} \to 0$, and equation (20) evaluates to

$$\sum_{z \in [a,b]} f(z) \cdot \Delta z \to \int_a^b f(z) dz$$

which is exactly what we wanted to show. □

Problem 2

Calculate the expectation and variance of the Poisson random variable based on the probability mass function.

Solution
Let X be a Poisson random variable with parameter . Then the PMF of X is given by:

$$P_X(k) = e^{-\lambda}\frac{\lambda^k}{k!}, \qquad \text{k} = 0,\, 1,\, \ldots \tag{21}$$

We can then use the PMF to calculate the expectation of X,

$$\begin{aligned}
\mathrm{E}[X] &= \sum_{k=0}^{\infty} k e^{-\lambda}\frac{\lambda^k}{k!} \\
&= \sum_{k=1}^{\infty} k e^{-\lambda}\frac{\lambda^k}{k!} \\
&= \lambda e^{-\lambda}\sum_{k=1}^{\infty}\frac{\lambda^{k-1}}{(k-1)!} \\
&= \lambda e^{-\lambda}\sum_{i=0}^{\infty}\frac{\lambda^{i}}{i!} \\
&= \lambda e^{-\lambda}(1 + \lambda + \frac{\lambda}{2!} + \cdots) \\
&= \lambda e^{-\lambda} e^{\lambda} \\
&= \lambda
\end{aligned}$$

We calculate $\mathrm{E}[X^2]$ as well as an intermediate step for calculating the variance of X.

$$\begin{aligned}
\mathrm{E}[X^2] &= \sum_{k=0}^{\infty} k^2 e^{-\lambda}\frac{\lambda^k}{k!} \\
&= \lambda e^{-\lambda}\sum_{k=1}^{\infty} k\frac{\lambda^{k-1}}{(k-1)!} \\
&= \lambda e^{-\lambda}\sum_{k=1}^{\infty}(k-1+1)\cdot\frac{\lambda^{k-1}}{(k-1)!} \\
&= \lambda e^{-\lambda}\left[\sum_{k=1}^{\infty}(k-1)\frac{\lambda^{k-1}}{(k-1)!} + \sum_{k=1}^{\infty}\frac{\lambda^{k-1}}{(k-1)!}\right] \\
&= \lambda e^{-\lambda}\left[\lambda\sum_{k=1}^{\infty}\frac{\lambda^{k-2}}{(k-2)!} + \sum_{j=0}^{\infty}\frac{\lambda^{j}}{j!}\right] \\
&= \lambda e^{-\lambda}\left[\lambda\sum_{i=0}^{\infty}\frac{\lambda^{i}}{i!} + e^{\lambda}\right] \\
&= \lambda e^{-\lambda}\left(\lambda e^{\lambda} + e^{\lambda}\right) \\
&= \lambda^2 + \lambda
\end{aligned}$$

Then the variance of X is given by

$$\mathrm{Var}(X) = \mathrm{E}[X^2] - \mathrm{E}[X]^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

□