STATS 200A: Homework #2

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Assignment: 1-5

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Problem 1

For a continuous random variable $X \sim f(x)$, prove

1. $\mathrm{E}[h(X) + g(X)] = \mathrm{E}[h(X)] + \mathrm{E}[g(X)]$.
2. $\mathrm{Var}(X) = \mathrm{E}[X^2] = \mathrm{E}[X]^2$
3. $\mathrm{E}[aX + b] = a\mathrm{E}[X] + b$ and $\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$
4. Let $\mathrm{E}[X] = \mu$ and $\mathrm{Var}(X) = \sigma^2$. Let $Z = (X - \mu)/\sigma$. Calculate $\mathrm{E}[Z]$ and $\mathrm{Var}(Z)$.

Solution
(1) By the definition of the expectation of a continuous random variable with and using linearity of integrals, we can express the left hand side as

$$\begin{aligned}
\mathrm{E}[h(X) + g(X)] &= \int_{-\infty}^{\infty} (h(x) + g(x)) f(x) dx \\
&= \int_{-\infty}^{\infty} h(x) f(x) dx + \int_{-\infty}^{\infty} g(x) f(x) dx \\
&= \mathrm{E}[h(X)] + \mathrm{E}[g(X)]
\end{aligned}$$

(2) By the definition of the variance of a continuous random variable and using linearity of expectation, we can express the left hand side as

$$\begin{aligned}
\mathrm{Var}(X) &= \mathrm{E}[(X - \mathrm{E}[X])^2] \\
&= \mathrm{E}[X^2 - 2X\mathrm{E}[X] + \mathrm{E}[X]^2] \\
&= \mathrm{E}[X^2] + 2\mathrm{E}[X]\mathrm{E}[X] + \mathrm{E}[X]^2 \\
&= \mathrm{E}[X^2] - \mathrm{E}[X]^2 + \mathrm{E}[X]^2 \\
&= \mathrm{E}[X^2] - \mathrm{E}[X]^2
\end{aligned}$$

(3) By the definition of the expectation of a continuous random variable, using linearity of integrals and the normalization axiom, we can express the left hand side as

$$\begin{aligned}
\int_{-\infty}^{\infty} (ax + b) f(x) dx &= \int_{-\infty}^{\infty} axf(x) dx + b \int_{-\infty}^{\infty} f(x) dx \\
&= a \int_{-\infty}^{\infty} xf(x) dx + b \cdot 1 \\
&= a\mathrm{E}[X] + b
\end{aligned}$$

For the second equality, we again use linearity of expectation

$$\begin{aligned}
\mathrm{Var}(aX+b) &= \mathrm{E}[(aX+b-\mathrm{E}[aX+b])^2] \\
&= \mathrm{E}[(aX+b-a\mathrm{E}[X]-b)^2] \\
&= \mathrm{E}[(aX-a\mathrm{E}[X])^2] \\
&= \mathrm{E}[a^2(X-\mathrm{E}[X])^2] \\
&= a^2\mathrm{E}[(X-\mathrm{E}[X])^2] \\
&= a^2\mathrm{Var}(X)
\end{aligned}$$

(4) Using the property proved in (3) above, we can express the left hand side as

$$\begin{aligned}
\mathrm{E}(Z) &= \mathrm{E}\left(\frac{X-\mu}{\sigma}\right) \\
&= \mathrm{E}\left(\frac{X}{\sigma}-\frac{\mu}{\sigma}\right) \\
&= \mathrm{E}\left(\frac{X}{\sigma}\right)-\mathrm{E}\left(\frac{\mu}{\sigma}\right) \\
&= \frac{1}{\sigma}\mathrm{E}(X)-\frac{\mu}{\sigma} \\
&= \frac{\mu}{\sigma}-\frac{\mu}{\sigma} \\
&= 0
\end{aligned}$$

For the second equality, we also use the property of the variance proved in (3) above,

$$\begin{aligned}
\mathrm{Var}(Z) &= \mathrm{Var}\left(\frac{X-\mu}{\sigma}\right) \\
&= \mathrm{Var}\left(\frac{X}{\sigma}-\frac{\mu}{\sigma}\right) \\
&= \frac{1}{\sigma^2}\mathrm{Var}(X) \\
&= \frac{1}{\sigma^2}\cdot\sigma^2 \\
&= 1
\end{aligned}$$

All equalities above are thus justified. $\square$

Problem 2

Let $U \sim \text{Unif}[0, 1]$, i.e., the density of T is $f(t) = \lambda e^{-\lambda t}$ for $t \geq 0$, $f(t) = 0$ for $t < 0$.

1. Calculate $F(u) = P(U \leq u)$
2. Calculate $\text{E}[U], \text{E}[U^2]$, and $\text{Var}(U)$.

Solution

Problem 3

Let $T \sim \text{Exp}(\lambda)$, i.e. the density of T is $f(t) = \lambda e^{-\lambda t}$ for $t \geq 0$ and $f(t) = 0$ for $t < 0$.

1. Calculate $F(t) = P(T \leq t)$. Find t so that $F(t) = 1/2$.
2. Calculate $\text{E}[T], \text{E}[T^2], \text{Var}(T)$

Solution

Problem 4

Let $Z \sim \mathrm{N}(0,1)$, i.e., the density of Z is

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

1. Calculate $\mathrm{E}[Z], \mathrm{E}[Z^2], \mathrm{Var}(Z)$.
2. Let $X = \mu + \sigma Z$, $\sigma > 0$. Find the density of X. Calculate $\mathrm{E}[X]$ and $\mathrm{Var}(X)$.

Solution

Problem 5

If $X \sim f_X(x)$.

1. Find the density of $Y = aX + b$.
2. Find the density of $Z = X^2$

You can either use cumulative density functions or work with probability density functions.

Solution