STATS 200A: Homework #1

Professor Yingnian Wu
Assignment: 1-3

Eric Chuu
UID: 604406828

October 6, 2016

Problem 1

Suppose we flip a fair coin independently n times.

1. What is the sample space Ω? How large is it?
2. For each sequence $\omega \in \Omega$, let $X_i(\omega) = 1$ if the i-th flip of ω is head and $X_i(\omega) = 0$ otherwise, $i = 1, \ldots, n$. For small $\varepsilon > 0$, let $A = \{w : \left|\sum_{i=1}^n X_i(\omega)/n - \frac{1}{2}\right| < \varepsilon\}$. Explain that
$$P(A) = P\left(\left|\sum_{i=1}^n X_i/n - 1/2\right| < \varepsilon\right) = |A|/|\Omega| \tag{1}$$
3. Explain the weak law of large numbers.

Solution
(1) If we flip a fair coin independently n times, the sample space Ω consists of the sequences of length n:

$$\begin{aligned}\Omega &= \{\text{sequences of length } n \text{ of the form:}(HHH\cdots H), (HHH\cdots T), \ldots, (TTT\cdots T)\} \\ &= \{H, T\}^n\end{aligned}$$

The cardinality of the sample space Ω is 2^n. □

(2) To explain that the equality in equation (1) above holds, we first note that since we are given a probability law, we know that the axioms of nonnegativity, additivity, and normalization hold and that to every event A, a number $P(A)$ is assigned. Moreover, since the coin is fair, all outcomes are equally likely. In other words, each $\omega \in \Omega$ is equally likely to occur. By the normalization property, we then have

$$P(\omega_1) + P(\omega_2) + \cdots + P(\omega_n) = n \cdot \frac{1}{n} = 1$$

By the second equality, we can view each $\omega_i \in \Omega$, $i = 1, \ldots n$, as a point within the sample space. Since A is nothing but a subset of these points, we can then calculate the probability of the event A as

$$P(A) = \frac{\text{points in A}}{n} = \frac{|A|}{|\Omega|}$$

which is exactly the equality shown in equation (1). □

(3) The **Weak Law of Large Numbers** is stated as follows:

Let X be a real-valued random variable, and let $X_1, X_2, \ldots$ be an infinite sequence of i.i.d. with $\mathrm{E}[X_i] = \mu$ for $i = 1, 2, \ldots$, where μ is finite. Let $\overline{X_n} := \frac{1}{n}(X_1 + \cdots + X_n)$ be the empirical average of this sequence. Then $\overline{X_n}$ converges in probability to μ, where this notion of convergence in probability can be expressed as: for every $\varepsilon > 0$,

$$\lim_{n \to \infty} P(\left|\overline{X_n} - \mu\right| \geq \varepsilon) = 0$$

Intuitively, this means that the sample mean converges in probability towards the expected value. In other words, for any nonzero difference, with a sample large enough, there is a high probability that the difference between the empirical average and the expected value of said sample is smaller in magnitude than the previously established nonzero difference. $\square$

Problem 2

Suppose we flip a fair coin independently infinitely many times.

1. What is the sample space Ω?
2. What is the $\sigma-$algebra? You need only explain
   (a) What are the basic events (statements)
   (b) How to build up the σ-algebra based on the basic events?
3. For each sequence $\omega \in \Omega$, let $X_i(\omega) = 1$ if the i-th flip of ω is head and $X_i(\omega) = 0$ otherwise. For a small $\varepsilon > 0$, let $A_n = \{\omega : |\sum_{i=1}^n X_i(\omega)/n - 1/2| < \varepsilon\}$. Explain the strong law of large numbers in terms of A_n.

Solution

(1) The sample space Ω for flipping a fair coin independently infinitely many times is

$$\begin{aligned}\Omega &= \{\text{infinite sequences with terms from } \{H, T\}\} \\ &= \{H, T\}^\infty\end{aligned}$$

□

(2) (a) The *basic events* are statements about the the first n tosses, where $n < \infty$. In other words, the basic events are statements about outcomes that can be determined by looking at the result of the first n tosses.

(b) To build up the σ-algebra based on the basic events defined above, we define $\mathcal{F}$ as the collection of countably many disjoint events. Since its an algebra, it is also closed under unions, complements, and thus intersections. Since we want $\mathcal{F}$ to be a σ-algebra, we also allow $\mathcal{F}$ to be closed under the combination of countably infinite many events A_i. □

(3) We first state the **strong law of large numbers** and then explain it in terms of A_n as defined above.

Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with mean μ. Then, the sequence of sample means converges to μ with probability 1:

$$P\left(\lim_{n\to\infty} \sum_{i=1}^n X_i/n = \mu\right) = 1$$

In the context of this coin flipping problem $\mu = 1/2$, and if we define the event A,

$$A := \{\omega : \lim_{n\to\infty} \sum_{i=1}^n X_i/n = 1/2\},$$

then we can express A in terms of A_n:

$$A = \bigcap_{\epsilon>0} \bigcup_{N=1}^\infty \bigcap_{n=N}^\infty A_n,$$

where $\epsilon \in \{1/k, k = 1, 2, \ldots\}$. In other words, by the strong law of large numbers, for any given $\epsilon > 0$, the difference $|\sum_{i=1}^n X_i/n - \mu|$ is greater than ϵ finitely many times. □

Problem 3

For a continuous random variable $X\tilde{f}(x)$, prove

1. $\mathrm{E}[h(X) + g(X)] = \mathrm{E}[h(X)] + \mathrm{E}[g(X)]$.
2. $\mathrm{Var}(X) = \mathrm{E}[X^2] = \mathrm{E}[X]^2$
3. $\mathrm{E}[aX + b] = a\mathrm{E}[X] + b$ and $\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$
4. Let $\mathrm{E}[X] = \mu$ and $\mathrm{Var}(X) = \sigma^2$. Let $Z = (X - \mu)/\sigma$. Calculate $\mathrm{E}[Z]$ and $\mathrm{Var}(Z)$.

Solution

(1) By the definition of the expectation of a continuous random variable with and using linearity of integrals, we can express the left hand side as

$$\begin{aligned}
\mathrm{E}[h(X) + g(X)] &= \int_{-\infty}^{\infty} (h(x) + g(x))f(x)dx \\
&= \int_{-\infty}^{\infty} h(x)f(x)dx + \int_{-\infty}^{\infty} g(x)f(x)dx \\
&= \mathrm{E}[h(X)] + \mathrm{E}[g(X)]
\end{aligned}$$

(2) By the definition of the variance of a continuous random variable and using linearity of expectation, we can express the left hand side as

$$\begin{aligned}
\mathrm{Var}(X) &= \mathrm{E}[(X - \mathrm{E}[X])^2] \\
&= \mathrm{E}[X^2 - 2X\mathrm{E}[X] + \mathrm{E}[X]^2] \\
&= \mathrm{E}[X^2] + 2\mathrm{E}[X]\mathrm{E}[X] + \mathrm{E}[X]^2 \\
&= \mathrm{E}[X^2] - \mathrm{E}[X]^2 + \mathrm{E}[X]^2 \\
&= \mathrm{E}[X^2] - \mathrm{E}[X]^2
\end{aligned}$$

(3) By the definition of the expectation of a continuous random variable, using linearity of integrals and the normalization axiom, we can express the left hand side as

$$\begin{aligned}
\int_{-\infty}^{\infty} (ax + b)f(x)dx &= \int_{-\infty}^{\infty} axf(x)dx + b\int_{-\infty}^{\infty} f(x)dx \\
&= a\int_{-\infty}^{\infty} xf(x)dx + b \cdot 1 \\
&= a\mathrm{E}[X]b
\end{aligned}$$

For the second equality, we again use linearity of expectation

$$\begin{aligned}
\mathrm{Var}(aX + b) &= \mathrm{E}[(aX + b - \mathrm{E}[aX + b])^2] \\
&= \mathrm{E}[(aX + b - a\mathrm{E}[X] - b)^2] \\
&= \mathrm{E}[(aX - a\mathrm{E}[X])^2] \\
&= \mathrm{E}[a^2(X - \mathrm{E}[X])^2] \\
&= a^2\mathrm{E}[(X - \mathrm{E}[X])^2] \\
&= a^2\mathrm{Var}(X)
\end{aligned}$$

(4) Using the property proved in (3) above, we can express the left hand side as

$$
\begin{aligned}
\mathrm{E}(Z) &= \mathrm{E}\left(\frac{X-\mu}{\sigma}\right) \\
&= \mathrm{E}\left(\frac{X}{\sigma} - \frac{\mu}{\sigma}\right) \\
&= \mathrm{E}\left(\frac{X}{\sigma}\right) - \mathrm{E}\left(\frac{\mu}{\sigma}\right) \\
&= \frac{1}{\sigma}\mathrm{E}(X) - \frac{\mu}{\sigma} \\
&= \frac{\mu}{\sigma} - \frac{\mu}{\sigma} \\
&= 0
\end{aligned}
$$

For the second equality, we also use the property of the variance proved in (3) above,

$$
\begin{aligned}
\mathrm{Var}(Z) &= \mathrm{Var}\left(\frac{X-\mu}{\sigma}\right) \\
&= \mathrm{Var}\left(\frac{X}{\sigma} - \frac{\mu}{\sigma}\right) \\
&= \frac{1}{\sigma^2}\mathrm{Var}(X) \\
&= \frac{1}{\sigma^2} \cdot \sigma^2 \\
&= 1
\end{aligned}
$$

All equalities above are thus justified. $\square$