MATH 170A: Homework #5

Professor P.F. Rodriguez

Lecture 1

Assignment: 1 - 6; Textbook: 3, 4, 9, 10; Supplementary: 4

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Eric Chuu

UID: 604406828

Problem 1

Let X be a Binomial random variable with parameters 50 and 0.2. What is the probability of the event that $X \leq 5$?

Solution
We're given that $X \sim Binom(50, 0.2)$. For $X \leq 5$, we need to calculate the PMF, $p_X(k)$ for $k = 0, \cdots, 5$.

$$\mathbf{P}(X \leq 5) = \sum_{k=0}^{5} \binom{50}{k} 0.2^k \cdot 0.8^{50-k}$$

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Problem 2

Let a, b, n be positive integers such that $n \leq a$ and $n \leq b$. Construct a probabilistic model (that is describe a random experiment) and a random variable X whose probability mass function is

$$p_X(k) = \frac{\binom{a}{k}\binom{b}{n-k}}{\binom{a+b}{n}},$$

for $k = 0, 1, \cdots, n$. The random experiment should involve red and black balls inside an urn.

Solution
Since we can take $|\Omega| = \binom{a+b}{n}$, then if we let a be the number of red balls, b be the number of black balls, then $|\Omega|$ is the number of ways to draw n balls from $a+b$ total balls from an urn. Then the numerator of the PMF is the number of ways to draw exactly k red balls from the a red balls and $n-k$ black balls from the b black balls. In this case, X is a discrete random variable that represents the number of red balls picked from the urn of $a+b$ total balls. ■

Problem 3

If X is Bernoulli random variable with parameter p, show that $Y = 1 - X$ is also a Bernoulli random variable. What is its parameter?

Solution
Since $X \sim Ber(p)$, its PMF is given by

$$p_X(x) = \begin{cases} p & \text{for x} = 1 \\ 1-p & \text{for } x = 0 \end{cases}$$

Since $Y = 1 - X$, it takes on different real values depending on the values that X takes on. Thus, for $x = 1$, $Y = 1 - 1 = 0$, and for $x = 0$, $Y = 1 - 0 = 1$. Thus, the PMF of Y is

$$p_Y(y) = \begin{cases} 1-p & \text{for y} = 1 \\ p & \text{for } y = 0 \end{cases}$$

so Y is Bernoulli random variable. Calculating the expectation of X gives us the parameter p, so similarly, we calculate the expectation of Y,

$$\mathrm{E}(Y) = \sum_{k=0}^{\infty} k \cdot P_Y(k) = 0 \cdot p + 1 \cdot (1-p) = (1-p)$$

Then $Y \sim Ber(1-p)$. ■

Problem 4

If X is Binomial random variable with parameters n and p, show that $Y = n - X$ is also a Binomial random variable. What are the parameters of Y?

Solution
Since $X \sim Binom(n, p)$, then for a fixed value of n, $Y = n - X$ depends on the values that X takes on, so for $x \in X$, we have $x \in \{0, 1, \cdots, n\}$, so for $y \in Y$, $y \in \{0, 1, \cdots, n\}$, so Y can similarly be viewed as the number of successes (or failures) in n tosses, so Y is a Binomial random variable. Since the expectation of X, $\mathrm{E}(X) = n \cdot p$, which is the product of the parameters of X, we can find the parameters of Y in a similar way. Calculating expectation of Y and using linearity of expectation, we get

$$\mathrm{E}(Y) = \mathrm{E}(n - X) = \mathrm{E}(n) - \mathrm{E}(X) = n - n \cdot p = n \cdot (1-p).$$

Then $Y \sim Binom(n, 1-p)$. ■

Problem 5

Let X be a Binomial random variable with paramenters n and $1/2$. Assume that n is odd. Find the PMF of the random variable $Y = X \bmod 2$. Show that the PMF does not depend on n.

Solution
We're given $X \sim (n, 1/2)$. Note that the PMF of X is

$$p_X(k) = \binom{n}{k} \cdot \left(\frac{1}{2}\right)^n$$

for $k = 0, 1, \cdots, n$. For $y \in Y$, $y = 0$ when X is even, and $y = 1$ when X is odd. Then we calculate the PMF of Y.

$$p_Y(y) = \begin{cases} \binom{n}{0}\left(\frac{1}{2}\right)^n + \binom{n}{2}\left(\frac{1}{2}\right)^n + \cdots + \binom{n}{n-1}\left(\frac{1}{2}\right)^n & \text{for } y = 0 \\ \\ \binom{n}{1}\left(\frac{1}{2}\right)^n + \binom{n}{3}\left(\frac{1}{2}\right)^n + \cdots + \binom{n}{n}\left(\frac{1}{2}\right)^n & \text{for } y = 1 \end{cases}$$

Given that $\binom{n}{k} = \binom{n}{n-k}$, we can write $\mathbf{P}(Y = 0)$ as

$$\begin{aligned} \mathbf{P}(Y = 0) &= \binom{n}{0}\left(\frac{1}{2}\right)^n + \binom{n}{2}\left(\frac{1}{2}\right)^n + \cdots + \binom{n}{n-1}\left(\frac{1}{2}\right)^n \\ &= \binom{n}{n-0}\left(\frac{1}{2}\right)^n + \binom{n}{n-2}\left(\frac{1}{2}\right)^n + \cdots + \binom{n}{n-(n-1)}\left(\frac{1}{2}\right)^n \\ &= \binom{n}{n}\left(\frac{1}{2}\right)^n + \binom{n}{n-2}\left(\frac{1}{2}\right)^n + \cdots + \binom{n}{1}\left(\frac{1}{2}\right)^n \\ &= \mathbf{P}(Y = 1) \end{aligned}$$

Thus, we've shown that $\mathbf{P}(Y = 0) = \mathbf{P}(Y = 1)$, so the PMF of Y does not depend on n. ■

Problem 6

In a soccer tournament you are playing once with each of the other nine teams. For each match, you get 3 points if you win, 1 point for a draw, a 0 points if you lose. For each match,

$$\begin{aligned} \mathbf{P}(\text{win}) &= 0.5 \\ \mathbf{P}(\text{draw}) &= 0.2 \\ \mathbf{P}(\text{lose}) &= 0.3, \end{aligned} \tag{1}$$

independently of the results of all other matches. What is the probability that you finish the tournament with at least 20 points?

Solution

We consider separately the events of having at least 20 points and exactly k wins, where $k \in \{0, 1, \cdots, 9\}$. The minimum number of games that you can win and still have at least 20 points is 6 games. There are two cases when you win 6 games and get 20 points, both shown below with their probabilities:

$$\text{6 wins, 2 draws, 1 loss with probability: } \frac{9!}{6! \cdot 2!} \cdot 0.5^6 \cdot 0.2^2 \cdot 0.3$$
$$\text{6 wins, 3 draws with probability: } \frac{9!}{6! \cdot 3!} \cdot 0.5^6 \cdot 0.2^3$$

Similarly, there are three cases when you win 7 games and get 20 points,

$$\text{7 wins, 2 draws with probability: } \frac{9!}{7!2!} \cdot 0.5^7 \cdot 0.2^2$$
$$\text{7 wins, 1 draw, 1 loss with probability: } \frac{9!}{7!} \cdot 0.5^7 \cdot 0.2 \cdot 0.3$$
$$\text{7 wins, 2 losses with probability: } \frac{9!}{7!2!} \cdot 0.5^7 \cdot 0.3^2$$

There are two cases when you win 8 games and get 20 points,

$$\text{8 wins, 1 draw with probability: } \frac{9!}{8!} \cdot 0.5^8 \cdot 0.2$$
$$\text{8 wins, 1 loss with probability: } \frac{9!}{8!} \cdot 0.5^8 \cdot 0.3$$

Finally, there is the case when you 9 games,

$$\text{9 wins with probability: } \frac{9!}{9!} \cdot 0.5^9$$

Then the probability of finishing the tournament with at least 20 points is given by

$$\frac{9!}{6! \cdot 2!} \cdot 0.5^6 \cdot 0.2^2 \cdot 0.3 + \frac{9!}{6! \cdot 3!} \cdot 0.5^6 \cdot 0.2^3 + \frac{9!}{7!2!} \cdot 0.5^7 \cdot 0.2^2 + \frac{9!}{7!} \cdot 0.5^7 \cdot 0.2 \cdot 0.3 +$$
$$\frac{9!}{7!2!} \cdot 0.5^7 \cdot 0.3^2 + \frac{9!}{8!} \cdot 0.5^8 \cdot 0.2 + \frac{9!}{8!} \cdot 0.5^8 \cdot 0.3 + \frac{9!}{9!} \cdot 0.5^9.$$

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Problem 7

(a) What is the probability that Fischer wins the match?

Solution
Let A be the event that Fisher wins the game, B be the event that Spassky wins, and C be the event that they draw, and W be the event that Fisher wins the match, with probability $\mathbf{P}(A) = 0.4, \mathbf{P}(B) = 0.3, \mathbf{P}(C) = 0.3$. Fisher wins if he wins the first game, if he draws the first game and wins the second game, etc. More generally, he wins the game if he draws in all the games preceding a win. This can be expressed as

$$\mathbf{P}(W) = \sum_{n=1}^{10} \mathbf{P}(A)\mathbf{P}(C)^{n-1} = \sum_{n=1}^{10} 0.4 \cdot 0.3^{n-1}$$

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(b) What is the PMF of the duration of the match?

Solution
Let X be the duration of the match, or equivalently, the number of turns in the match. Then $x \in X$ takes on values $x \in \{1, \cdots, 10\}$. The match lasts 10 turns when both players draw for 9 consecutive games, and this occurs with probability 0.3^9. For match durations between 1 and 9, the match ends when either player wins, which occurs with probability 0.7. Then X follows a geometric distribution, so

$$p_X(x) = \begin{cases} 0.3^{x-1} \cdot (0.7) & \text{for } x = 1, \cdots, 9 \\ 0.3^9 & \text{for } x = 10 \\ 0 & \text{otherwise} \end{cases}$$

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Problem 8

(a) What is the PMF of the number of modems in use at the given time?

Solution

Let X be the number of modems in use at any time. We consider the values that $x \in X$ can take on: $x \in \{1, \cdots, 50\}$. We're given that the probability that a modem is in use is 0.01, independent of other customers. Since X is a Bernoulli random variable, we can express the PMF as

$$p_X(x) = \binom{1000}{x} \cdot 0.01^x \cdot 0.99^{1000-x}$$

for $x = 0, \cdots, 49$. Note that the probability when $x = 50$, that is, when 50 modems are in use, is the same as the probability that 50 or more people of the 1000 need to use a modem. Thus, for $x = 50, \cdots, 1000$,

$$p_X(x) = \mathbf{P}(X = 50) = \sum_{x=50}^{1000} 0.01^x \cdot 0.99^{1000-x}$$

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(b) Repeat part (a) by approximating the PMF of the number of customers that need a connection with a Poisson PMF.

Solution

We take $Y = X$, X defined as above, with $p = 0.01, n = 1000$, so $\lambda = np = 10$. So $Y \sim Poi(\lambda)$. Then we can estimate the PMF of the number of customers that need a connection with the Poisson PMF,

$$p_Y(k) = e^{-10} \cdot \frac{10^k}{k!}$$

for $k = 0, 1, \cdots 49$. For $k \geq 50$,

$$p_Y(k) = \sum_{k=50}^{1000} e^{-10} \cdot \frac{10^k}{k!}$$

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(c) What is the probability that there are more customers needing a connection than there are modems?

Let $A = \{\text{More than 50 customers need connection}\}$. With X defined as above, it follows a binomial distribution, then we can calculate the probability,

$$\mathbf{P}(A) = \sum_{k=51}^{1000} \binom{1000}{k} \cdot 0.01^k \cdot 0.99^{1000-k}$$

Using the Poisson approximation from part (b), we have

$$\mathbf{P}(A) \approx \sum_{k=51}^{1000} e^{-10} \cdot \frac{10^k}{k!}$$

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Problem 9

Show that the PMF $p_X(k)$ is monotonically nondecreasing with k in the range 0 to k, and is monotonically decreasing with k for $k \geq k^*$.

Solution

We're given that $X \sim Binom(n,p)$ and k^* be the largest integer that is less than or equal to $(n+1)p$, so $k^* \leq (n+1)p$. Consider

$$\begin{aligned}\frac{p_X(k)}{p_X(k-1)} &= \frac{\binom{n}{k}p^k(1-p)^{n-k}}{\binom{n}{k-1}p^{k-1}(1-p)^{n-k+1}} \\ &= \frac{n-k+1}{k}\cdot p\cdot\frac{1}{1-p} \\ &= \frac{(n-k+1)p}{k(1-p)} \\ &= \frac{(n+1)p-kp}{k-kp}\end{aligned}$$

$p_X(k)$ is monotonic nondecreasing $\Leftrightarrow \frac{p_X(k)}{p_X(k-1)} \geq 1 \Leftrightarrow (n+1)p \geq k^* \geq k$. $p_X(k)$ is monotonic decreasing $\Leftrightarrow \frac{p_X(k)}{p_X(k-1)} < 1 \Leftrightarrow (n+1)p < k$, so $k \geq k^*$. Therefore, we've shown that for values k between 0 to k^*, $p_X(k)$ is monotonically nondecreasing, and for values $k \geq k^*$, $p_X(x)$ is monotonically decreasing. ■

Problem 10

Show that PMF $p_X(k)$ increases monotonically with k up to the point where k reaches the largest integer not exceeding λ, and after that point decreases monotonically with k.

Solution

We're given that $X \sim Poi(\lambda)$. Then consider

$$\begin{aligned}\frac{p_X(k)}{p_X(k-1)} &= \frac{\lambda/e^{\lambda}k!}{\lambda^{k-1}/e^{\lambda}(k-1)!} \\ &= \frac{\lambda}{k}\end{aligned}$$

$p_X(k)$ is monotonic increasing $\Leftrightarrow \frac{p_X(k)}{p_X(k-1)} \geq 1 \Leftrightarrow k \leq \lambda$. $p_X(k)$ is monotonic decreasing $\Leftrightarrow \frac{p_X(k)}{p_X(k-1)} < 1 \Leftrightarrow k > \lambda$. Thus, we've shown that the $p_X(k)$ is monotonic increasing for values of $k \leq \lambda$ and monotonic decreasing for values $k > \lambda$. ■

Problem 11

Let X be a discrete random variable, and let $Y = |X|$.

(a) Assume that the PMF of X is

$$p_X(x) = \begin{cases} Kx^2 & \text{if } x = -3, -2, -1, 0, 1, 2, 3 \\ 0 & \text{otherwise,} \end{cases}$$

where K is a suitable constant. Determine the value of K.

(b) For the PMF of X given in part (a) calculate the PMF of Y.

(c) Give a general formula for the PMF of Y in terms of the PMF of X.

Solution
Since $\sum_x p_X(x) = 1$, we see that $K(2 \cdot 3^2 + 2 \cdot 2^2 + 2 \cdot 1^1) = 1$, so $K = \frac{1}{14}$. Thus, the PMF of X can be rewritten

$$p_X(x) = \begin{cases} \frac{1}{28}x^2 & \text{if } x = -3, -2, -1, 0, 1, 2, 3 \\ 0 & \text{otherwise} \end{cases}$$

We first consider the possible values of $y \in Y$. $Y = |X|$, so $y \in \{0, 1, 2, 3\}$. Since both -1 and 1 get mapped to 1, $y = 1$ occurs with twice the probability than it did when $X = 1$. Then, the PMF of Y is given by

$$p_Y(y) = \begin{cases} \frac{2}{28}x^2 & \text{if } y = 0, 1, 2, 3 \\ 0 & \text{otherwise} \end{cases}$$

As mentioned before, the values of y occur with sum of the probabilities of the values of x that result in y. Thus, we can express the PMF of Y in terms of the PMF of X with

$$p_Y(y) = \sum_{\{x:|x|=y\}} P_X(x).$$

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