MATH 170A: Homework #2

Professor P.F. Rodriguez
Lecture 1
Assignment: 1, 4, 5; Chapter 1: 14, 18, 54, 56; Supplemental: 1, 2, 3, 5, 34, 36
January 19, 2016

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Problem 1

For which values of k would you advise him to withdraw and for which values of k would you advise him to stay in the game.

Solution

Let A be the event in which he wins and B be the event in which the higher of his two cards is k. In doing so, we can calculate the conditional probability of him winning the game given that the value of his higher card is k:

$$\mathbf{P}(A|B) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(B)} = \frac{(|A \cap B|)}{|B|}.$$

The number of ways in which k is his higher card, $|B|$, is given by $(k-1) \cdot \binom{18}{2}$, where $(k-1)$ represents the number of ways to choose the other card in his hand, and $\binom{18}{2}$ represents the number of ways for the other player to choose 2 cards from the remaining 18 possible cards. Then the number of ways in which he wins the game and his higher card is k, $|A \cap B|$, is given by $(k-1) \cdot \binom{k-2}{2}$, where $(k-1)$ represents the number of ways to choose the other card that is less than k, and $\binom{k-2}{2}$ represents the number of ways for the other player to choose his two cards. The two cards are chosen from $(k-2)$ cards because we exclude the other card the first player picks as his lower card. Evaluating the conditional probability, we get

$$\mathbf{P}(A|B) = \frac{(k-2) \cdot \binom{k-2}{2}}{(k-1) \cdot \binom{18}{2}} = \frac{(k-2)(k-3)}{18 \cdot 17}. \tag{1}$$

To determine for which values of k he should withdraw and which values he should stay in the game, we find values of k that result in the probability given in (1) being greater than $\frac{1}{2}$.

$$\frac{(k-2)(k-3)}{306} > \frac{1}{2}$$
$$k^2 - 5k - 147 > 0$$

We find that for values $k = -9.8784$ and $k = 14.879$, we get a probability that is greater than $\frac{1}{2}$, but since k is a number between 1 and 20, we conclude that for $k \in [15, 20]$, he should stay in the game, while for $k \in [2, 14]$, he should withdraw from the game. ■

Problem 2

On a lottery ticket you choose and circle six numbers out of the numbers $1, 2, 3, \ldots 44, 45$. The next day at the lottery six numbers are chosen at random (with all combinations having the same probability) out of the numbers $1, 2, 3, \ldots 44, 45$. You win the prize if on your lottery ticket you have guessed exactly three out of this six numbers. What is the probability you win the prize?

Solution

Let A be the event that we win the lottery. That is, we choose exactly three numbers correctly and exactly three numbers incorrectly, but in no particular order. The number of ways to choose correct numbers is given by: $\binom{6}{3}$. This leaves 39 other incorrect numbers, so the number of ways to pick the wrong numbers

Problem 2 continued on next page. . .

is given by $\binom{39}{3}$. Hence there are $\binom{6}{3} \cdot \binom{39}{3}$ ways to win the lottery. To caluclate the probability, we consider Ω, which is the set of all possibleoutcomes when picking 6 numbers from 45, so $|\Omega| = \binom{45}{6}$. We calculate the probability of winning the lottery

$$\mathbf{P}(A) = \frac{|A|}{|\Omega|} = \frac{\binom{6}{3} \cdot \binom{39}{3}}{\binom{45}{6}}$$

■

Problem 3

You have an 8×8 chessboard and a token at the low left corner square. You want to move it to the top right corner square. In the questions below you need to compute in how many ways you can do that given the restrictions, that is how many such token trajectories exists.

(a) In how many ways can you move the token if, at each move you're only allowed to move the token one square to the right or one square up?

Solution
We're essentially counting the number of ways we can arrange the 7 upward and 7 rightward movements that will get the token to the top corner square. Since the 7 upward movements are not distinct from each other and the 7 rightward movements are not distinct from each other, the total number of distinct paths can be represented by $\frac{14!}{7! \cdot 7!}$. ■

(b) In how many ways can you move the token if in addition you're also allowed to move the token diagonally?

Solution
We can count the number of ways we can move the token by splitting the movements into cases that differ in the number of diagonal movements allowed, which we represent with k, where k can range from 0 diagonal steps, which is essentially part (a), to 7 diagonal steps, which consists of getting from the lower left to the upper right using only diagonal steps. The cases and their corresponding number of outcomes are as follows:
Case $k = 0$: 7 ups, 7 rights: $\frac{14!}{7! \cdot 7!}$
Case $k = 1$: 1 diagonal, 6 ups, 6 rights: $\frac{13!}{6! \cdot 6!}$
Case $k = 2$: 2 diagonals, 5 ups, 5 rights: $\frac{12!}{5! \cdot 5! \cdot 2!}$
Case $k = 3$: 3 diagonals, 4 ups, 4 rights: $\frac{11!}{4! \cdot 4! \cdot 3!}$
Case $k = 4$: 4 diagonals, 3 ups, 3 rights: $\frac{10!}{3! \cdot 3! \cdot 4!}$
Case $k = 5$: 5 diagonals, 2 ups, 2 rights: $\frac{9!}{2! \cdot 2! \cdot 5!}$
Case $k = 6$: 6 diagonals, 1 up, 1 right: $\frac{8!}{6!}$
Case $k = 7$: 7 diagonals, 0 ups, 0 rights: $\frac{7!}{7!}$

Hence the total number of ways we can move when allowing diagonal movements is given by summing up the number of outcomes corresponding to each of the above cases:

$$\frac{14!}{7! \cdot 7!} + \frac{13!}{6! \cdot 6!} + \frac{12!}{5! \cdot 5! \cdot 2!} + \frac{11!}{4! \cdot 4! \cdot 3!} + \frac{10!}{3! \cdot 3! \cdot 4!} + \frac{9!}{2! \cdot 2! \cdot 5!} + \frac{8!}{6!} + \frac{7!}{7!}$$

■

Problem 4

We roll two fair 6-sided dice. Each one of the 36 possible outcomes is assumed equally likely.

(a) Find the probability that doubles are rolled.

Solution
Let A be the event that doubles are rolled. Then $A = \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\}$. Then

$$\mathbf{P}(A) = \frac{|A|}{|\Omega|} = \frac{6}{36} = \frac{1}{6}$$

The probability that doubles are rolled is $\frac{1}{6}$. ■

(b) Given the roll results in a sum of 4 or less, find the conditional probability that doubles are rolled.

Solution
Let A be the event that doubles are rolled and B be the event that the roll results in a sum of 4 or less. Then

$$A = \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\}$$
$$B = \{(1,1),(1,2),(1,3),(2,1),(2,2),(3,1)\}$$

Then

$$\mathbf{P}(A|B) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(B)} = \frac{2/36}{6/36} = \frac{1}{3}$$

Thus the conditional probability that doubles are rolled given the roll results in a sum of 4 or less is $\frac{1}{3}$. ■

(c) Find the probability that at least one die roll is a 6.

Solution
Let A be the event that at least one die roll is a 6. Then

$$A = \{(1,6),(2,6),(3,6),(4,6),(5,6),(6,6),(6,5),(6,4),(6,3),(6,2),(6,1)\}$$
$$\mathbf{P}(A) = \frac{11}{36}.$$

The probability that at least one die roll is a 6 is $\frac{11}{36}$. ■

(d) Given that the two dice land on different numbers, find the conditional probability that at least one die roll is a 6.

Solution
Let A be the event that at least one die roll is a 6 and B be the event that the two dice land on different numbers. Then

$$B^c = \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\}$$
$$\mathbf{P}(B) = 1 - \mathbf{P}(B^c) = 1 - \frac{6}{36} = \frac{30}{36}$$

Using our results from part (c) to evaluate $\mathbf{P}(A \cap B)$, we can then evaluate the conditional probability

$$\mathbf{P}(A|B) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(B)} = \frac{10/36}{30/36} = \frac{1}{3}$$

The conditional probability that at least one die roll is a 6 given that the two dice land on different numbers is $\frac{1}{3}$. ■

Problem 5

Let A and B be events. Show that $\mathbf{P}(A \cap B|B) = \mathbf{P}(A|B)$, assuming that $\mathbf{P}(B) > 0$.

Solution
We know that RHS $= \mathbf{P}(A|B) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(B)}$, so evaluating the LHS, we get

$$\text{LHS} = \mathbf{P}(A \cap B|B) = \frac{\mathbf{P}((A \cap B) \cap B)}{\mathbf{P}(B)} = \frac{\mathbf{P}(A \cap (B \cap B))}{\mathbf{P}(B)} = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(B)} = \text{RHS}.$$

Hence, the equality is proven. ■

Problem 6

(a) In how many different ways can the cars line up?

Solution
The cars can line up in 20! different ways. ■

(b) What is the probability that on a given day, the cars will park in such a way that they alternate (no two US-made are adjacent and no two foreign-made are adjacent)?

Solution
Let A be the event that the cars park in alternate order as described above. We first compute the number of ways to place the 10 US cars, which is 10!, followed by the number of ways to place the 10 foreign cars, which is 10!. Combining these two arrangements we get $10! \cdot 10!$, but since either type of car can be placed "first" in the parking lot, we multiply this number by 2. The sample space is the result of part (a). Then the probability is given by

$$\mathbf{P}(A) = \frac{|A|}{|\Omega|} = \frac{2 \cdot 10! \cdot 10!}{20!}$$

The probability that the cars are alternately parked is $\frac{2 \cdot 10! \cdot 10!}{20!}$ ■

Problem 7

A valid curriculum consists of 4 lower level courses and 3 higher level courses.

(a) How many different curricula are possible?

Solution
Let L be the set of all lower level coures and H be the set of all higher level coures. Then $L = \{L_1, \cdots, L_8\}$ and $H = \{H_1, \cdots, H_{10}\}$. Then $|L| = 8, |H| = 10$. The total number of different curricula is then given by $\binom{8}{4} \cdot \binom{10}{3}$ ■

(b) Suppose that that $\{H_1, \cdots, H_5\}$ have L_1 as a prerequisite, and $\{H_6, \cdots, H_{10}\}$ have L_2 and L_3 as prerequisites. How many different curricula are there?

Solution
In order to count the total number of different curricula, we must first split the curricula into cases, with

each case having at least one of the three lower level courses: L_1, L_2, L_3. Consider:

Case 1: We don't pick L_1 but we pick both L_2, L_3. Then from the remaining 5 lower level courses, we pick 2. From the 5 possible higher level courses, we pick 3. Thus, there are $\binom{5}{2} \cdot \binom{5}{3}$ different curricula.

Case 2: We pick L_1 but we don't pick neither L_2 nor L_3. Then there are $\binom{5}{3} \cdot \binom{5}{3}$ possible curricula.

Case 3: We pick L_1 and either L_2 or L_3 but not both. Then there are $2 \cdot \binom{5}{2} \cdot \binom{5}{3}$ possible curricula. We multiply by 2 to account for the case where we pick L_2 and the case where we pick L_3, both of which yield $\binom{5}{2} \cdot \binom{5}{3}$.

Case 4: We pick L_1 and both L_2 and L_3. Then there are $\binom{5}{1} \cdot \binom{10}{3}$ possible curricula.

To find the total number of different curricula, we add the possible curricula corresponding to each of these cases: $\binom{5}{2} \cdot \binom{5}{3} + \binom{5}{3} \cdot \binom{5}{3} + 2 \cdot \binom{5}{2} \cdot \binom{5}{3} + \binom{5}{1} \cdot \binom{10}{3}$. ∎

Problem 8

We are given that $\mathbf{P}(A) = 0.55$, $\mathbf{P}(B^c) = 0.35$, $\mathbf{P}(A \cup B) = 0.75$. Determine $\mathbf{P}(B)$ and $\mathbf{P}(A \cap B)$.

Solution
We can calculate $\mathbf{P}(B) = 1 - \mathbf{P}(B^c) = 1 - 0.35 = 0.65$. Then

$$\begin{aligned} \mathbf{P}(A \cap B) &= \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cup B) \\ &= 0.55 + 0.65 - 0.75 \\ &= 0.45 \end{aligned}$$

Hence $\mathbf{P}(B) = 0.65$ and $\mathbf{P}(A \cap B) = 0.45$. ∎

Problem 9

Let A, B be two sets. Under what conditions is the set $A \cap (A \cup B)^c$ empty?

Solution
If $B \subset A$, then $A \cup B = A$, and $A \cap (A \cup B)^c = A \cap (A)^c = \emptyset$. ∎

Problem 10

Let A, B be two sets.

(a) Show that $(A^c \cap B^c)^c = A \cup B$ and $(A^c \cup B^c)^c = A \cap B$.

Solution
By De Morgan's Law, we see that $(A^c \cap B^c)^c = ((A^c)^c \cup (B^c)^c) = (A \cup B)$. The second equality is proven similarly. $(A^c \cup B^c)^c = ((A^c)^c \cap (B^c)^c) = A \cap B$. ∎

(b) Consider rolling a six-sided die once. Let A be the set of outcomes where an odd number comes up. Let B be the set of outcomes where a 1 or a 2 comes up. Calculate the sets on both sides of the equalities in part (a), and verify that the equalities hold.

Solution
$A = \{1, 3, 5\}, B = \{1, 2\}$. Then evaluating the LHS of the first equality from part (a), we see that

$$\begin{aligned} \text{LHS} &= (A^c \cap B^c)^c \\ &= (\{2, 4, 6\} \cap \{3, 4, 5, 6\})^c \\ &= (\{4, 6\})^c \\ &= \{1, 2, 3, 5\} \end{aligned}$$

Then evaluating the RHS, we see that

$$\begin{aligned} \text{RHS} &= \{1, 3, 5\} \cup \{1, 2\} \\ &= \{1, 2, 3, 5\} \\ &= \text{LHS}. \end{aligned}$$

We've shown that the first equality holds. Consider the LHS of the second equality:

$$\begin{aligned} \text{LHS} &= ((\{1, 3, 5\})^c \cup (\{1, 2\})^c)^c \\ &= ((\{2, 4, 6\}) \cup (\{3, 4, 5, 6\}))^c \\ &= (\{2, 3, 4, 5, 6\})^c \\ &= \{1\} \end{aligned}$$

Consider the RHS:

$$\text{RHS} = \{1, 3, 5\} \cap \{1, 2\} = \{1\} = \text{LHS}$$

Thus, both equalities have been verified. ∎

Problem 11

We are given that $\mathbf{P}(A^c) = 0.6, \mathbf{P}(B) = 0.3, \mathbf{P}(A \cap B) = 0.2$. Determine $\mathbf{P}(A \cup B)$.

Solution
Since $\mathbf{P}(A) = 1 - \mathbf{P}(A^c) = 1 - 0.6 = 0.4$, we can calculate $\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cup B) = 0.4 + 0.3 - 0.2 = 0.5$. ∎

Problem 12

Consider three independent rolls of of a fair six-sided die.

(a) What is the probability that the sum of the three rolls is 11?

Solution
Let A be the event that the sum of the three rolls is 11.
Then $A =$

$$\begin{aligned}
&\{(1,6,4),(1,5,5),(1,4,6),(2,6,3),(2,5,4),(2,4,5),(2,3,6),\\
&(3,6,2),(3,5,3),(3,4,4),(3,3,5),(3,2,6),(4,6,1),(4,5,2),\\
&(4,4,3),(4,3,4),(4,2,5),(4,1,6),(5,5,1),(5,4,2),(5,3,3),\\
&(5,2,4),(5,1,5),(6,4,1),(6,3,2),(6,2,3),(6,1,4)\}
\end{aligned}$$

so $|A| = 27$ and $|\Omega| = 6^3$. Then

$$\mathbf{P}(A) = \frac{|A|}{|\Omega|} = \frac{27}{6^3}.$$

■

(b) What is the probability that the sum of the three rolls is 12?

Solution
Let A be the event that the sum of the three rolls is 12. $A =$

$$\begin{aligned}
&\{(1,6,5),(1,5,6),(2,6,4),(2,5,5),(2,4,6),(3,6,3),(3,5,4),(3,4,5),(3,3,6),\\
&(4,6,2),(4,5,3),(4,4,4),(4,3,5),(4,2,6),(5,6,1),(5,5,2),(5,4,3),(5,3,4),\\
&(5,2,5),(5,1,6),(6,5,1),(6,4,2),(6,3,3),(6,2,4),(6,1,5)\}
\end{aligned}$$

We calculate

$$\mathbf{P}(A) = \frac{25}{6^3}.$$

■

(c) In the seventeenth century, Galileo explained the experimental observation that a sum of 10 is more frequent than a sum of 9, even though both 10 and 9 can be obtained in six distinct ways.

Solution
We consider the six distinct ways in which we can obtain each sum. For a sum of 10, the distinct cases are: $\{(1,6,3),(1,5,4),(2,6,2),(2,5,3),(2,4,4),(3,4,3)\}$, with the corresponding different ways/permutations these cases can appear:

$$3! + 3! + \frac{3!}{2!} + 3! + \frac{3!}{2!} + \frac{3!}{2!} = 27.$$

For a sum of 9, we have the six cases: $\{(1,6,2),(1,5,3),(1,4,4),(2,5,2),(2,4,3),(3,3,3)\}$, with the corresponding permutations for each of the cases:

$$3! + 3! + \frac{3!}{2!} + \frac{3!}{2!} + 3! + \frac{3!}{3!} = 25.$$

As seen, although there are six distinct ways in which we can obtain the sums 9 and 10, there are 27 permutations that sum up to 10, while there are 25 permutations that sum up to 9, which supports Galileo's claim that the sum of 10 is more frequent than a sum of 9. ■

Problem 13

Count the number of of distinguishable ways in which you can arrange the letters in the words:

(a) children

Solution
Since the word children has 8 distinct characters, we can calculate the number of distinguishable ways to arrange the letters in the words as: 8! ■

(b) bookkeeper

Solution
We first calculate the permutations of the 10 letter word, so we get 10!, but since there are repeeated characters in the word, this way of counting overcounts the number of distinguishable ways, so we readjust this value to get a total of:

$$\frac{10!}{2! \cdot 2! \cdot 3!}$$

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