MATH 170A: Homework #1

Professor P.F. Rodriguez

Assignment: 1, 2, 3; Textbook Chapter 1: 2, 5, 6, 7, 8, 9, 10

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Problem 1

Solution
We first write $A \cup B$ as a disjoint union, so that when we take the probability, we can make use of the additivity axiom, as seen in (1) below.

$$\begin{aligned} A \cup B &= A \cup (A^c \cap B) \\ \mathbf{P}(A \cup B) &= \mathbf{P}(A \cup (A^c \cap B)) \\ &= \mathbf{P}(A) + \mathbf{P}(A^c \cap B) \end{aligned} \tag{1}$$

By the non-negativity axiom, $\mathbf{P}(A^c \cap B) \geq 0$, so it's clear that $\mathbf{P}(A) \leq \mathbf{P}(A \cup B)$.

Consider $A = (A \cap B^c) \cup (A \cap B)$, which is a disjoint union. Then taking the probability and applying the additivity axiom, we get:

$$\begin{aligned} \mathbf{P}(A) &= \mathbf{P}((B^c \cap A) \cup (A \cap B)) \\ &= \mathbf{P}(B^c \cap A) + \mathbf{P}(A \cap B) \end{aligned} \tag{2}$$

By non-negativity, $\mathbf{P}(B^c \cap A) \geq 0$, so $\mathbf{P}(A \cap B) \leq \mathbf{P}(A)$. Thus, by transitivity, we get

$$\mathbf{P}(A \cap B) \leq \mathbf{P}(A) \leq \mathbf{P}(A \cup B).$$

■

Problem 2

Solution
We're given that $\Omega = \{1, 2, 3, 4, 5, 6\}$, and that $\mathbf{P}(\{1, 2, 3, 4\}) = 0.6, \mathbf{P}(\{4\}) = 0.2, \mathbf{P}(\{3, 4, 5, 6\}) = 0.9$. From this, we can have:

$$\begin{aligned} &\mathbf{P}(\{1\}) = \mathbf{P}(\{2\}) = 0.05 \\ &\mathbf{P}(\{3\}) = 0.3 \\ &\mathbf{P}(\{4\}) = \mathbf{P}(\{5\}) = \mathbf{P}(\{6\}) = 0.2 \end{aligned}$$

Alternatively, we can have a probability law with the following:

$$\begin{aligned} &\mathbf{P}(\{1\}) = \mathbf{P}(\{2\}) = 0.05 \\ &\mathbf{P}(\{3\}) = 0.3 \\ &\mathbf{P}(\{4\}) = 0.2 \\ &\mathbf{P}(\{5\}) = 0.1 \\ &\mathbf{P}(\{6\}) = 0.3 \end{aligned}$$

Both of these satisfy the information given, so it's not enough to determine a unique probability law. ■

Problem 3

Solution
We're given that there are 3 model types, 2 engine types, 2 transmission types, and 5 colors. If the car is hybrid, then it must also have an automatic transmission. Thus, the number of ways to configure a normal car is given by: $3*1*2*5 = 30$, while the number of ways to configure a hybrid car is given by $3*1*1*5 = 15$, resulting in a total of 45 ways to configure a car. ■

Problem 4

Let A, B be two sets.

(a) Show that $A^c = (A^c \cap B) \cup (A^c \cap B^c)$ and $B^c = (A \cap B^c) \cup (A^c \cap B^c)$.

Solution
For any sets X, Y we can write $X = (X \cap Y) \cup (X \cap Y^c)$, a union of two disjoint set. If we then let $X = A^c, Y = B$, then

$$A^c = (A^c \cap B) \cup (A^c \cap B^c),$$

which yields the desired result. By interchanging the variables and letting $X = B^c, Y = A$, we get

$$\begin{aligned} B^c &= (B^c \cap A) \cup (B^c \cap A^c) \\ &= (A \cap B^c) \cup (A^c \cap B^c), \end{aligned}$$

which proves the second equality. ■

(b) Show that: $(A \cap B)^c = (A^c \cap B) \cup (A^c \cap B^c) \cup (A \cap B^c)$.

Solution
By De Morgan's law and applying the results of part (a), we can write,

$$\begin{aligned} (A \cap B)^c &= A^c \cup B^c \\ &= (A^c \cap B) \cup (A^c \cap B^c) \cup (A \cap B^c) \cup (A^c \cap B^c) \\ &= (A^c \cap B) \cup (A^c \cap B^c) \cup (A \cap B^c) \end{aligned}$$

We can leave off the recurring $A^c \cap B^c$ because they represent the same set. ■

(c) Consider rolling a fair six-sided die. Let A be the set of outcomes where the roll is an odd number. Let B be the set of outcomes where the roll is less than 4. Calculate the sets on both sides of the equality in part (b), and verify that the equality holds.

Solution
We consider the quantities separately:

$$\begin{aligned} (A \cap B)^c &= \{\text{set of outcomes where roll is not (1 or 3)}\} \\ (A^c \cap B) &= \{\text{set of outcomes that are 2}\} \\ (A^c \cap B^c) &= \{\text{even numbers } \geq 4\} \\ (A \cap B^c) &= \{\text{odd numbers } \geq 4\} \end{aligned}$$

Looking at the right hand side of the equality from part (b), we see that

$$\begin{aligned} (A^c \cap B) \cup (A^c \cap B^c) \cup (A \cap B^c) &= \{2\} \cup \{4, 6\} \cup \{5\} \\ &= \{2, 4, 5, 6\} \\ &= \{\text{set of outcomes where roll is not (1 or 3)}\} \\ &= \text{LHS} \end{aligned}$$

thus satisfying the equality from part (b). ■

Problem 5

Out of the students in a class, 60% are geniuses, 70% love chocolate, and 40% fall into both categories. Determine the probability that a randomly selected student is neither a genius nor a chocolate lover.

Solution

Let A = genius, B = chocolate lover, so $\mathbf{P}(A) = 0.6, \mathbf{P}(B) = 0.7, \mathbf{P}(A \cap B) = 0.4$. Then $(A^c \cap B^c) =$ {event in which student is neither genius nor chocolate lover}. Taking the probability and applying De Morgan's law, we get

$$\begin{aligned}\mathbf{P}(A^c \cap B^c) &= \mathbf{P}(A \cup B)^c \\ &= 1 - \mathbf{P}(A \cup B) \\ &= 1 - (\mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B)) \\ &= 1 - (0.6 + 0.7 - 0.4) \\ &= 0.1.\end{aligned}$$

Thus, the probability that the randomly selected student is neither a genius nor a chocolate lover is 0.1 ■

Problem 6

A six-sided die is loaded in a way that each even face is twice as likely as each odd face. All even faces are equally likely, as are all odd faces. Construct a probabilistic model for a single roll of this die and find the probability that the outcome is less than 4.

Solution

We first specify the sample space $\Omega = \{1, 2, 3, 4, 5, 6\}$, all of which are disjoint. Then the probabilities are as follows:

$$\begin{aligned}\mathbf{P}(1) = \mathbf{P}(3) = \mathbf{P}(5) &= \frac{1}{9} \\ \mathbf{P}(2) = \mathbf{P}(4) = \mathbf{P}(6) &= \frac{2}{9}\end{aligned}$$

Let $A = \{\text{outcome less than } 4\}$. Taking the probability of this event and using the additivity axiom, we get

$$\begin{aligned}\mathbf{P}(A) = \mathbf{P}(1 \cup 2 \cup 3) &= \mathbf{P}(1) + \mathbf{P}(2) + \mathbf{P}(3) \\ &= \frac{1}{9} + \frac{2}{9} + \frac{1}{9} \\ &= \frac{4}{9},\end{aligned}$$

so the probability of getting an outcome of a dice roll less than 4 is $\frac{4}{9}$. ■

Problem 7

A four-sided die is rolled repeatedly, until the first time (if ever) that an even number is obtained. What is the sample space for this experiment?

Solution
There are 2 cases and thus 2 sample spaces to consider:

Case 1: A *finite* sequence for which an even number is rolled on the nth roll. Then,

$$x_1, x_2, ..., x_{n-1} \in \{1, 3\},$$
$$x_n \in \{2, 4\}.$$

In this case, the sample space consists of all possible outcomes of the above specified elements. Note that these are sequences of length n.

Case 2: An *infinite* sequence for which an even number roll never occurs. Then,

$$x_1, x_2, ... \in \{1, 3\}.$$

In this case, the sample space consists of all possible outcomes of the above specified elements, consisting entirely of odd numbers, either 1 or 3. Note that the elements of this sample space are infinite sequences. ■

Problem 8

You enter a special kind of chess tournament, in which you play one game with each of three opponents, but you get to choose the order in which you play your opponents, knowing the probability of a win against each. You win the tournament if you win two games in a row, and you want to maximize the probability of winning. Show that it is optimal to play the weakest opponent second, and that the order of playing the other two opponents does not matter.

Solution
We first find the probability of winning the tournament. Winning the tournament involves winning the 2nd game because we need to win 2 consecutive games. Thus, there are three possible events that result in winning the tournament: $\{WWW\}$, $\{WWL\}$, and $\{LWW\}$, where a W represents a win and an L represents a loss. Foe example, WWL is the event that we win the first 2 rounds but lose the 3rd round. Let $\mathbf{P}(\text{win first round}) = p1$, $\mathbf{P}(\text{win second round}) = p2$, and $\mathbf{P}(\text{win third round}) = p3$. Since WWW, WWL, and LWW are disjoint, we can write the probability of winning the tournament as:

$$\begin{aligned}\mathbf{P}(\{WWW, WWL, LWW\}) &= \mathbf{P}(WWW) + \mathbf{P}(WWL) + \mathbf{P}(LWW) \\ &= p1 \cdot p2 \cdot p3 + p1 \cdot p2 \cdot (1 - p3) + (1 - p1) \cdot p2 \cdot p3 \\ &= p2(p1 + p3 - p1 \cdot p3).\end{aligned}$$

We then claim that playing the opponents in the order (1,2,3) with corresponding probabilities $p1, p2, p3$ optimizes the probability of winning the tournament. This means the probability calculated above must be greater than or equal to the probability of winning if we choose alternative orders of play. Hence, if $p2 \geq p1$ and we compare the new probability of winning the tournament if we play against the player who we have $p1$ probability of beating in the 2nd round with the probability of our claimed optimal probability, we see that

$$p2(p1 + p3 - p1 \cdot p3) \geq p1(p2 + p3 - p2 \cdot p3).$$

Problem 8 continued on next page...

Note the in parentheses on the LHS, $p1, p3$ are reflexive, so the order in which we play the remaining two players does not matter. The same can be said about the probabilities in the parentheses on the RHS as well. Comparing the probability of winning the tournament if we play the player against whom we have $p3$ probability of winning, we see that

$$p2(p1 + p3 - p1 \cdot p3) \geq p3(p1 + p2 - p1 \cdot p2).$$

In both cases, we see that playing against the player against whom we have the highest probability of winning in the 2nd round optimizes our probability of winning the tournament. The reflexive nature of the other probabilities in the parentheses seen in the above inequalities show that the order that we play the other players does not matter. ■

Problem 9

A partition of the sample space Ω is a collection of disjoint events $S_1, ..., S_n$ such that $\Omega = \cup_{i=1}^n S_i$.

(a) Show that for any event A, we have

$$\mathbf{P}(A) = \sum_{i=1}^{n} \mathbf{P}(A \cap S_i).$$

Solution

We can express $A = \cup_{i=1}^n (A \cap S_i)$. Since the events S_i are disjoint and $(A \cap S_i) \subset S_i$ for $i = 1, ..., n$, then $A \cap S_i$ for $i = 1, ..., n$ are also disjoint. Then taking the probability, we get

$$\begin{aligned}\mathbf{P}(A) &= \mathbf{P}(\cup_{i=1}^n (A \cap S_i)) \\ &= \mathbf{P}(A \cap S_1) + ... + \mathbf{P}(A \cap S_n) \\ &= \sum_{i=1}^{n} \mathbf{P}(A \cap S_i),\end{aligned}$$

hence the desired result. ■

(b) Use part (a) to show that for any events A, B, C, we have

$$\mathbf{P}(A) = P(A \cap B) + \mathbf{P}(A \cap C) + \mathbf{P}(A \cap B^c \cap C^c) - \mathbf{P}(A \cap B \cap C).$$

Solution

Consider the sets $(B^c \cap C^c), (B \cap C), (B^c \cap C), (B \cap C^c)$, which form a partition of the sample space Ω. Then by part (a),

$$\mathbf{P}(A) = \mathbf{P}(A \cap B^c \cap C^c) + \mathbf{P}(A \cap B \cap C) + \mathbf{P}(A \cap B^c \cap C) + \mathbf{P}(A \cap B \cap C^c) \tag{3}$$

Note that $A \cap B = (A \cap B \cap C) \cup (A \cap B \cap C^c)$, a disjoint union. Taking the probability and applying the additivity axiom, we get

$$\mathbf{P}(A \cap B) = \mathbf{P}(A \cap B \cap C) + \mathbf{P}(A \cap B \cap C^c). \tag{4}$$

Similarly, $A \cap C = (A \cap B \cap C) \cup (A \cap B^c \cap C)$. Taking the probability and applying additivity, we get

$$\mathbf{P}(A \cap C) = \mathbf{P}(A \cap B \cap C) + \mathbf{P}(A \cap B^c \cap C). \tag{5}$$

Rearranging and substituting (4) and (5) back into (3), we get

$$\begin{aligned}\mathbf{P}(A) &= \mathbf{P}(A \cap B^c \cap C^c) + \mathbf{P}(A \cap B) - \mathbf{P}(A \cap B \cap C^c) + \mathbf{P}(A \cap C) - \mathbf{P}(A \cap B \cap C) + \mathbf{P}(A \cap B \cap C^c) \\ &= \mathbf{P}(A \cap B^c \cap C^c) + \mathbf{P}(A \cap B) + \mathbf{P}(A \cap C) - \mathbf{P}(A \cap B \cap C),\end{aligned}$$

which is exactly the equality we are trying to prove. ■

Problem 10

Show that the formula

$$\mathbf{P}((A \cap B^c) \cup (A^c \cap B)) = \mathbf{P}(A) + \mathbf{P}(B) - 2\mathbf{P}(A \cap B).$$

Solution
We first note that on the LHS, $(A \cap B^c) \cup (A^c \cap B)$ is a disjoint union since $(A \cap B^c) \subset A$ and $(A^c \cap B) \subset A^c$, so using additivity, we can rewrite the LHS to get:

$$\begin{aligned}\text{LHS} &= \mathbf{P}((A \cap B^c) \cup (A^c \cap B)) \\ &= \mathbf{P}(A \cap B^c) + \mathbf{P}(A^c \cap B)\end{aligned}$$

We know that $\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B)$, so the RHS can be written,

$$\begin{aligned}\text{RHS} &= \mathbf{P}(A) + \mathbf{P}(B) - 2\mathbf{P}(A \cap B) \\ &= \mathbf{P}(A \cup B) + \mathbf{P}(A \cap B) - 2\mathbf{P}(A \cap B) \\ &= \mathbf{P}(A \cup B) - \mathbf{P}(A \cap B)\end{aligned}$$

We can write $B = (A \cap B) \cup (A^c \cap B)$, which is a disjoint union, so evaluating the RHS,

$$\begin{aligned}\text{RHS} &= \mathbf{P}(A \cup B) - \mathbf{P}(A \cap B) \\ &= \mathbf{P}(A \cup (A \cap B) \cup (A^c \cap B)) - \mathbf{P}(A \cap B) \\ &= \mathbf{P}(A) \cup (A^c \cap B)) - \mathbf{P}(A \cap B) \\ &= \mathbf{P}(A) + \mathbf{P}(A^c \cap B) - \mathbf{P}(A \cap B)\end{aligned}$$

We can write $A = (A \cap B) \cup (B^c \cap A)$. Substituting this back into the RHS, we get

$$\begin{aligned}\text{RHS} &= \mathbf{P}((A \cap B) \cup (B^c \cap A)) + \mathbf{P}(A^c \cap B) - \mathbf{P}(A \cap B) \\ &= \mathbf{P}(A \cap B) + \mathbf{P}(B^c \cap A) + \mathbf{P}(A^c \cap B) - \mathbf{P}(A \cap B) \\ &= \mathbf{P}(B^c \cap A) + \mathbf{P}(A^c \cap B) \\ &= \text{LHS},\end{aligned}$$

so the equality is proven. ■