# ST2132
AY23/24 SEM 1
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# 01. PROBABILITY

- **probability** of an event → the limiting relative frequency of its occurrence as the experiment is repeated many times
- the **realisation** $x$ is a constant, and $X$ is a generator
  - running $r$ experiments gives us $r$ realisations $x_1, \dots, x_r$

## Expectation

**discrete**: (mass function)
$$E(X) := \sum_{i=1}^{n} x_i p_i$$

**continuous**: (density function)
$$E(X) := \int_{-\infty}^{\infty} x f(x)\, dx$$

### expectation of a function $h(X)$

$$E\{h(X)\} = \begin{cases} \sum_{i=1}^{n} h(x_i) p_i & X \text{ is discrete} \\ \int_{-\infty}^{\infty} h(x) f(x)\, dx & X \text{ is continuous} \end{cases}$$

## Variance

**variance**, $\text{var}(X) := E\{(X-\mu)^2\}$

**standard deviation**, $SD(X) := \sqrt{\text{var}(X)}$

- $\text{var}(X) = E(X^2) - E(X)^2$
- $E(X - \mu) = 0$

## Law of Large Numbers

mean and variance of $r$ realisations:
$$\bar{x} := \frac{1}{r}\sum_{i=1}^{r} x_i \qquad v := \frac{1}{r}\sum_{i=1}^{r}(x_i - \bar{x})^2$$

**LLN:** for a function $h$, as $r \to \infty$,
$$\frac{1}{r}\sum_{i=1}^{r} h(x_i) \to E\{h(X)\}$$
$$\bar{x} \to E(X), \quad v \to \text{var}(X)$$

### Monte Carlo approximation

simulate $x_1, \dots, x_r$ from $X$. by LLN, as $r \to \infty$, the approximation becomes exact
$$E\{h(X)\} \approx \frac{1}{r}\sum_{i=1}^{r} h(x_i)$$

## Joint Distribution

(**discrete**) mass function:
$$P(X = x_i, Y = y_j) = p_{ij}$$
(**continuous**) density function:
$$f: \mathbb{R}^2 \to [0, \infty), \int_{-\infty}^{\infty} f(x,y)\, dx\, dy = 1$$
(**expectation**) for $h: \mathbb{R}^2 \to \mathbb{R}$,
$$E\{h(X,Y)\} = \begin{cases} \sum_{i=1}^{I}\sum_{j=1}^{J} h(x_i, y_j) p_{ij} & X \text{ is discrete} \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y) f(x,y)\, dx\, dy & Y \text{ is continuous} \end{cases}$$

## Algebra of RV's

let $X, Y$ be RVs and $a, b, c$ be constants
- $Z = aX + bY + c$ is also an RV
  - $z = ax + by + c$ is a realisation of $Z$
- linearity of expectation: $E(Z) = aE(X) + bE(Y) + c$
- any theorem about a RV is true about a constant

## Covariance

let $\mu_X = E(X), \mu_Y = E(Y)$.

**covariance**, $\text{cov}(X,Y) = E\{(X - \mu_X)(Y - \mu_Y)\}$
- $\text{cov}(X,Y) = E(XY) - \mu_X \mu_Y$
- $\text{cov}(X,Y) = \text{cov}(Y,X)$
- $\text{cov}(X,X) = \text{var}(X)$
- $\text{cov}(W, aX + bY + c) = a\,\text{cov}(W,X) + b\,\text{cov}(W,Y)$
- $\text{var}(aX + bY + c) =$ $a^2\,\text{var}(X) + b^2\,\text{var}(Y) + 2ab\,\text{cov}(X,Y)$
- $\text{var}(\sum_{i=1}^{N} a_i X_i) =$ $\sum_{i=1}^{N} a_i^2\,\text{var}(X_i) + 2\sum_{1 \le i < j \le N} a_i a_j\,\text{cov}(X_i, X_j)$

## joint $=$ marginal $\times$ conditional distributions

$$f(x,y) = f_X(x) f_Y(y|x)$$
$$= f_Y(y) f_X(x|y), \quad x, y \in \mathbb{R}$$

- $f(x,y)$ is the *joint density*
- $f_X(x), f_Y(y)$ are the *marginal densities*
- $f_Y(\cdot|x)$ is the **conditional** density of $Y$ given $X = x$
- $f_X(\cdot|y)$ is the **conditional** density of $X$ given $Y = y$
- for discrete case, *density* $\equiv$ *probability*, $x \equiv x_i$, $y \equiv y_j$

## Independence

- $X, Y$ are independent $\iff \forall x, y \in \mathbb{R}$,
  1. $f(x,y) = f_X(x) f_Y(y)$
  2. $f_Y(y|x) = f_Y(y)$
  3. $f_X(x|y) = f_Y(x)$
- $X, Y$ are independent $\Rightarrow$
  - $E(XY) = E(X)E(Y)$
  - $\text{cov}(X,Y) = 0$

  (the converse does not hold)

## Conditional expectation

### discrete case

let $f_Y(\cdot|x_i)$ be the conditional pmf of $Y$ given $X = x_i$.
$$E[Y|x_i] := \sum_{j=1}^{J} y_j f_Y(y_j|x_i)$$
$$\text{var}[Y|x_i] := \sum_{j=1}^{J} (y_j - E[Y|x_i])^2 f_Y(y_j|x_i)$$

$E[Y|x_i]$ is like $E(Y)$, with conditional distribution replacing marginal distribution $f_Y(\cdot)$. likewise, $\text{var}[Y|x_i]$ like $\text{var}(Y)$.

### continuous case

$$E[Y|x] := \int_{-\infty}^{\infty} y f_Y(y|x)\, dy$$
$$\text{var}[Y|x] := \int_{-\infty}^{\infty} (y - E[Y|x])^2 f_Y(y|x)\, dy$$
$$= E(Y^2|x) - \{E(Y|x)\}^2$$

## Distributions

if $X$ is iid with expectation $\mu$, SD $\sigma$ and $S_n = \sum_{i=1}^{n} X_i$,
- $E(S_n) = n\mu$
- $SD(S_n) = \sqrt{n}\sigma$
- variance of sum = sum of variances
  $\text{var}(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} \text{var}(x_i)$

### bernoulli

$X \sim Bernoulli(p)$ $\Rightarrow$ coin flip with probability $p$
$$E(X_i) = p \qquad \text{var}(X_i) = p(1-p)$$
$$E(S_n) = np \qquad \text{var}(S_n) = np(1-p)$$

### binomial

$X \sim Bin(n,p) \quad \Rightarrow X_i \overset{i.i.d.}{\sim} Bernoulli(p)$
$$E(X) = np, \quad \text{var}(X) = np(1-p)$$
$$E(X) = \sum_{k=1}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$$
$$\text{cov}(X, n - X) = -\text{var}(X)$$

### multinomial

$X \sim Multinomial(n, \mathbf{p})$
- for $k$ outcomes $E_1, \dots, E_k$, $Pr(E_i) = p_i$. For some $1 \le i \le k$, $E_i$ occurs $X_i$ times in $n$ runs.

$(X_1, \dots, X_k)$ has the **multinomial distribution:**
$$Pr(X_1 = x_1, \dots, X_k = x_k) = \binom{n}{x_1, \dots, x_k} \Pi_{i=1}^{k} p_i^{x_i}$$
- where $\binom{n}{x_1,\dots,x_k} = \frac{n!}{x_1! x_2! \dots x_k!}$
  - combinatorially, # of arrangements of $x_1, \dots, x_k$
  - $\sum_{i=1}^{n} x_i = n, \quad x_i \ge 0$

$$E(X) = \begin{bmatrix} np_1 \\ np_2 \\ \vdots \\ np_k \end{bmatrix}, \quad \text{var}(X_i) = np_i(1 - p_i)$$
$\text{var}(X) =$ *covariance matrix* $M$ with
$$m_{ij} = \begin{cases} \text{var}(X_i) & \text{if } i = j \\ \text{cov}(X_i, X_j) & \text{if } i \ne j \end{cases}$$
- $\text{cov}(X_i, X_j) < 0$
- $X_i \sim Bin(n, p_i)$
- $X_i + X_j \sim Bin(n, p_i + p_j)$

# 02. PROBABILITY (2)

## Mean Square Error (MSE)

$$MSE = E\{(Y - c)^2\}$$
- predicting Y:
  $MSE = \text{var}(Y) + \{E(Y) - c\}^2$
  - $\min MSE = \text{var}(Y)$ when $c = E(Y)$
- $Y$ and $X$ are correlated:
  $MSE = \text{var}[Y|x] + \{E[Y|x] - c\}^2$
  $MSE = E[(Y-c)^2|x] = E[\{Y - E(Y)\}^2|x]$
  - $\min MSE = \text{var}(Y|x)$ when $c = E[Y|x]$
  - if $c = E(Y)$ instead of $E(Y|x) \Rightarrow$ the MSE increases by $(E(Y|x) - E(Y))^2$

### mean MSE

$$\frac{1}{n}\sum_{i=1}^{n} \text{var}[Y|x_i] \approx E\{\text{var}[Y|X]\}$$

## random conditional expectations

let $X, Y$ be r.v.s.
- $E[Y|X]$ is a r.v. which takes value $E[Y|x]$ with probability/density $f_X(x)$
- $\text{var}[Y|X]$ is a r.v. which takes value $\text{var}[Y|x]$ with probability/density $f_X(x)$

$$E(E[X_2|X_1]) = E(X_2)$$
$$\text{var}(E[X_2|X_1]) + E(\text{var}[X_2|X_1]) = \text{var}(X_2)$$

## CDF (cumulative distribution function)

for r.v. $X$, let $F(x) = P(X \le x)$
- domain: $\mathbb{R}$; codomain: $[0,1]$
$$F(x) = \int_{-\infty}^{x} f(x)\, dx$$

## Standard Normal Distribution

$Z \sim N(0,1)$ has density function
$$\phi(z) = \frac{1}{\sqrt{2\pi}} \exp\{-\frac{z^2}{2}\}, \quad -\infty < z < \infty$$
$$E(Z) = 0, \quad \text{var}(Z) = 1$$
**CDF**, $\Phi(x) = P(Z \le x) = \int_{-\infty}^{x} \phi(z)\, dz$
- $E(Z) = \int_{-\infty}^{\infty} z\phi(z)\, dz = 0$
  - $E(Z^2) = \int_{-\infty}^{\infty} z^2 \phi(z)\, dz = 1$
  - $E(Z^{2k+1}) = 0 \quad \forall k \in \mathbb{Z}_{\ge 0}$

### general normal distribution

let $X \sim N(\mu, \sigma^2)$ and $Y \sim N(\nu, \tau^2)$

**standardisation:** $\frac{X - \mu}{\sigma} \sim N(0,1)$
- summations:
  - for constants $a, b \ne 0$,
    $a + bX \sim N(a + b\mu,\ b^2\sigma^2)$
  - $X + Y \sim N(\mu + \nu,\ \sigma^2 + \tau^2 + 2\,\text{cov}(X,Y))$
    - $\text{cov}(X,Y) = 0, \quad \Rightarrow \quad X \perp Y$
    - $X \perp Y \quad \Rightarrow \quad X + Y \sim N(\mu + \nu,\ \sigma^2 + \tau^2)$
- for $W = a + bX$,
  - density, $f_W(w) = \frac{d}{dw} F_W(w)$
  - CDF, $F_W(w) = P(X \le \frac{w-a}{b}) = \Phi(\frac{w-a}{b})$

## Central Limit Theorem

let $X_1, \dots, X_n$ be iid rv's with expectation $\mu$ and SD $\sigma$, with $S_n = \sum_{i=1}^{n} X_i$

**CLT**

as $n \to \infty$, the distribution of the standardised $S_n = \frac{S_n - n\mu}{\sqrt{n}\sigma}$ converges to $N(0,1)$

- $E(S_n) = n\mu$, $\text{var}(S_n) = n\sigma^2$
- for large $n$, approximately $S_n \sim N(n\mu, n\sigma^2)$

### bernoulli

let $X_i \sim Bernoulli(p)$. then $S_n \sim Binom(n,p)$
- for large $n$, $S_n \sim N(np, np(1-p))$
- CLT: standardised $\frac{S_n - np}{\sqrt{n}\sqrt{p(1-p)}} \to N(0,1)$ as $n \to \infty$

## Distributions

### chi-square ($\chi^2$)

let $Z \sim N(0,1)$. $\Rightarrow$ then $Z^2 \sim \chi^2_1$

- $Z^2$ has $\chi^2$ distribution with 1 degree of freedom
- degrees of freedom = number of RVs in the sum

$$E(Z^2) = 1, \quad E(Z^4) = 3$$
$$\text{var}(Z^2) = E(Z^4) - \{E(Z^2)\}^2 = 2$$

let $V_1, \ldots, V_n$ be iid $\chi^2_1$ RVs and $V = \sum_{i=1}^n V_i$. then

$$V \sim \chi^2_n$$
$$E(V) = n \quad \text{var}(V) = 2n$$

### gamma

let $\alpha, \lambda > 0$. The $Gamma(\alpha, \lambda)$ density is

$$\frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, \quad x > 0$$

where $\Gamma(\alpha)$ is a number that makes density integrate to 1

- $\chi^2_n$ RV $\sim Gamma(\frac{n}{2}, \frac{1}{2})$
  - $\chi^2_n$ is a special case of Gamma!
  - density of $\chi^2_1$ RV $= \frac{1}{\sqrt{2\pi}} v^{-1/2} e^{-v/2}, \quad v > 0$
    $= Gamma(\frac{1}{2}, \frac{1}{2})$
- if $X_1 \sim Gamma(\alpha_1, \lambda)$ and $X_2 \sim Gamma(\alpha_2, \lambda)$ are independent, then $X_1 + X_2 \sim Gamma(\alpha_1 + \alpha_2, \lambda)$

### t distribution

let $Z \sim N(0,1)$ and $V \sim \chi^2_n$ be independent.

$$\frac{Z}{\sqrt{V/n}} \sim t_n$$

has a $t$ distribution with $n$ degrees of freedom.

- $t$ distribution is symmetric around 0
- $t_n \to Z$ as $n \to \infty$ (because $\frac{V}{n} \to 1$)

### *F* distribution

let $V \sim \chi^2_m$ and $W \sim \chi^2_n$ be independent.

$$\frac{V/m}{W/n} \sim F_{m,n}$$

has an $F$ distribution with $(m, n)$ degrees of freedom.

- even if $m = n$, still two RVs $V, W$ as they are independent
- for $T \sim t_n$, $T^2 = \frac{Z^2}{V/n} \sim F_{1,n}$

### IID Random Variables

let $X_1, \ldots, X_n$ be iid RVs with mean $\bar{X}$.

**sample variance**, $S^2 = \dfrac{1}{n-1} \displaystyle\sum_{i=1}^n (X_i - \bar{X})^2$

$S$ is an estimate of $\sigma$

let $X_1, \ldots, X_n$ be iid $N(\mu, \sigma^2)$. $\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$.

$$\bar{X} \sim N(\mu, \tfrac{\sigma^2}{n})$$
$$E(\bar{X}) = \mu, \quad \text{var}(\bar{X}) = \frac{\sigma^2}{n}$$

more distributions:

$$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$$
$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$$
$$\frac{\bar{X} - \mu}{S/\sqrt{n}} \sim t_{n-1}$$

- $\bar{X}$ and $S^2$ are independent

## Multivariate Normal Distribution

let $\boldsymbol{\mu}$ be a $k \times 1$ vector and $\boldsymbol{\Sigma}$ be a *positive-definite* symmetric $k \times k$ matrix.

the random vector $\boldsymbol{X} = (X_1, \ldots, X_k)'$ has a multivariate normal distribution $N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ if its density function is

$$\frac{1}{(2\pi)^{k/2}\sqrt{det\boldsymbol{\Sigma}}} \exp\left(-\frac{(\boldsymbol{x} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{x} - \boldsymbol{\mu})}{2}\right)$$

- $E(\boldsymbol{X}) = \boldsymbol{\mu}, \quad \text{var}(\boldsymbol{X}) = \boldsymbol{\Sigma}$
- for any non-zero $k \times 1$ vector $\boldsymbol{a}$,
  $$\boldsymbol{a'X} \sim N(\boldsymbol{a'\mu}, \boldsymbol{a'\Sigma a})$$
  - $\boldsymbol{a'\Sigma a} > 0$ because $\boldsymbol{\Sigma}$ is positive-definite
  - the product $\boldsymbol{a'X}$ is a scalar (same for $\boldsymbol{a'\mu}, \boldsymbol{a'\Sigma a}$)
- two multinomial normal random vectors $\boldsymbol{X_1}$ and $\boldsymbol{X_2}$, sizes $h$ and $k$, are independent if $\text{cov}(\boldsymbol{X_1}, \boldsymbol{X_2}) = \boldsymbol{0}_{h \times k}$
  - $(X_1 - \bar{X}, \ldots, X_n - \bar{X})$ has a multivariate normal distribution; the covariance between $\bar{X}$ and $(X_1 - \bar{X}, \ldots, X_n - \bar{X})$ is $0$, thus they are independent

## 03. POINT ESTIMATION

for a variable $v$ in population $N$,

$$\mu = \frac{1}{N} \sum_{i=1}^N v_i \qquad \sigma^2 = \frac{1}{N} \sum_{i=1}^N (v_i - \mu)^2$$

- $\mu$, $\sigma^2$ are **parameters** (unknown constants)
- a **simple random sample** is used to estimate parameters: individuals drawn from the population at random without replacement

### binary variable

for variable $v$ with proportion $p$ in the population,

$$\mu = p, \qquad \sigma^2 = p(1-p)$$

### single random draw

for variable $v$ (population of size $N$, mean $\mu$, variance $\sigma^2$), let $X$ be the chosen $v$-value.

$$E(X) = \mu, \qquad \text{var}(X) = \sigma^2$$

### draws with replacement

let $X_1, \ldots, X_n$ be random draws with replacement from a population of mean $\mu$ and variance $\sigma^2$.

random sample mean, $\bar{X} = \dfrac{1}{n} \displaystyle\sum_{i=1}^n X_i$

$X_1, \ldots, X_n$ are iid with $E(X_i) = \mu$, $\text{var}(X_i) = \sigma^2$

$E(\bar{X}) = \mu$, $\text{var}(\bar{X}) = \frac{\sigma^2}{n}$

let $x_1, \ldots, x_n$ be realisations of $n$ random draws with replacement from the population.

sample mean, $\bar{x} = \dfrac{1}{n} \displaystyle\sum_{i=1}^n x_i$

- as $n \to \infty$, $\bar{x} \to \mu$ (LLN)
- sample distribution, $x_i$ has the same distribution as $X_i$ and the population distribution

### representativeness

- $X_1, \ldots, X_n$ is **representative** of the population
  - as $n$ gets larger, $\bar{X}$ gets closer to $\mu$
- $x_1, \ldots, x_n$ are *likely* representative of the population

## estimating mean

given data $x_1, \ldots, x_n$,

- sample mean, $\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$ is an **estimate** of $\mu$
- the error in $\bar{x}$ is $\mu - \bar{x}$; it cannot be estimated
- $\bar{x}$ is a realisation of the **estimator** $\bar{X}$
  - this realisation is used to estimate $\mu$

### standard error

the size of error in estimate $\bar{x}$ is roughly $SD(\bar{X}) = \dfrac{\sigma}{\sqrt{n}}$

the **standard error** (SE) in $\bar{x}$ is $\frac{\sigma}{\sqrt{n}}$

- SE is a constant by definition: $SE = SD(\bar{X}) = \frac{\sigma}{\sqrt{n}}$

### estimating $\sigma$

intuitive estimate of $\sigma^2$, $\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2$

sample variance, $s^2 = \dfrac{1}{n-1} \displaystyle\sum_{i=1}^n (x_i - \bar{x})^2$

$$E(s^2) = \sigma^2$$

### Point estimation of mean

a population (size $N$) has unknown mean $\mu$, variance $\sigma^2$. for random draws (without replacement) $x_1, \ldots, x_n$:

$\bar{x}$ is a realisation of $\bar{X}$, with $E(\bar{X}) = \mu$, $\text{var}(\bar{X}) = \frac{\sigma^2}{n}$

- $\mu$ is estimated as $\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$
- error in $\bar{x}$ is measured by the SE: $\frac{\sigma}{\sqrt{n}} = SD(\bar{X})$
- SE is estimated as $\frac{s}{\sqrt{n}}$

$\Rightarrow \mu$ is around $\bar{x}$, give or take $\frac{s}{\sqrt{n}}$

### unbiased estimation

- since $E(\bar{X}) = \mu$, $\bar{X}$ is an **unbiased** estimator of $\mu$. $\bar{x}$ is an unbiased estimate.
- $S^2$ is unbiased for $\sigma^2$: $E(S^2) = \sigma^2$
- $S$ is *not* unbiased for $\sigma$: $E(S) < \sigma$

### Simple random sampling (SRS)

$n$ random draws *without replacement* from a population of mean $\mu$ and variance $\sigma^2$.

- for $i = 1, \ldots, n$, $E(X_i) = \mu$ and $\text{var}(X_i) = \sigma^2$
- for $i \neq j$, $\text{cov}(X_i, X_j) = -\frac{\sigma^2}{N-1}$
- if $n/N$ is relatively large,
  - multiply SE by correction factor $\sqrt{\frac{N-n}{N-1}}$
  - standard error = $\frac{N-n}{N-1}$

$$E(\bar{X}) = \mu, \quad \text{var}(\bar{X}) = \frac{N-n}{N-1}\frac{\sigma^2}{n}$$

- if $n << N$, then SRS is like sampling *with replacement* (treat the data as if they come from IID RVs $X_1, \ldots, X_n$)

$$E(\bar{X}) = \mu, \quad \text{var}(\bar{X}) = \frac{\sigma^2}{n}$$

### estimating proportion $p$

- in a 0-1 population, $\mu = p$, $\sigma^2 = p(1-p)$
  - $p$ is estimated as $\bar{x}$ (sample proportion of 1's)
- $SE = \frac{\sqrt{p(1-p)}}{\sqrt{n}} = SD(\hat{p})$
  - estimated by replacing $p$ with $\bar{x}$
- unbiased estimator $\hat{p}$
  - $E(\hat{p}) = p, \quad \text{var}(\hat{p}) = \frac{p(1-p)}{n}, \quad SD(\hat{p}) = SE$

- the estimate of $\sigma$ is $\hat{\sigma}$, not $s$
- e.g. if a SRS of size 100 has 78 white balls,
  $p \approx 0.78 \pm \frac{\sqrt{0.78 \times 0.22}}{\sqrt{100}}$

### Gauss Model

Let $x_i$ be a realisation of $X_i$. $X_1, \ldots, X_{100}$ are random draws with replacement from an imaginary population with mean $w$ and variance $\sigma^2$. $w$ and $\sigma^2$ are parameters (unknown constants).

- $E(X_i) = w, \quad \text{var}\ X_i = \sigma^2$ (since $X_i$ is just 1 draw)
- $E(\bar{X}) = w, \quad \text{var}\ \bar{X} = \frac{\sigma^2}{100}$

## 04. ESTIMATION (SE, bias, MSE)

let $x_1, \ldots, x_n$ be from random draws $X_1, \ldots, X_n$ with replacement from a population of mean $\mu$ and variance $\sigma^2$.

sample mean $\bar{x}$ is an *unbiased estimate* of $\mu$

$$\bar{x} = \tfrac{1}{n} \textstyle\sum_{i=1}^n x_i$$

SE = $\frac{\sigma}{\sqrt{n}} \approx \frac{s}{\sqrt{n}}$ tells us roughly how far $\bar{x}$ is from $\mu$

sample variance, $s^2 = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2$

### MSE and bias

suppose measurements were from a population with mean $w + b$ where $b$ is a constant: $x_i = w + b + \epsilon_i$

- $E(\bar{X}) = w + b$
- $SD(\bar{X}) = \frac{\sigma}{\sqrt{n}}$
  - $SE = \frac{\sigma}{\sqrt{n}}$ measures how far $\bar{x}$ is from $w + b$, not $w$
- if $b \neq 0$, then $\bar{x}$ is a biased estimate for $w$

$$MSE = E\{(\bar{X} - w)^2\} = \frac{\sigma^2}{n} + b^2$$
$$MSE = SE^2 + bias^2$$

as $n \to \infty$, $MSE \to b^2$

### conclusion

let $\theta$ be a parameter (constant) and $\hat{\theta}$ be an estimator (RV).

$SE = SD(\hat{\theta})$, bias = $E(\hat{\theta}) - \theta$,

MSE = $E\{(\hat{\theta} - \theta)^2 = SE^2 + bias^2\}$

## 05. INTERVAL ESTIMATION

let $x_1, \ldots, x_n$ be realisations of IID RVs $X_1, \ldots, X_n$ with unknown $\mu = E(X_i)$ and $\sigma^2 = \text{var}(X_i)$.

sample mean, $\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$

sample variance, $s^2 = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2$

standard error, $SE = \frac{s}{\sqrt{n}}$

**point estimation:** $\mu \approx \bar{x}$, give or take $\frac{s}{\sqrt{n}}$

**interval estimation:** interval contains $\mu$ with some confidence level

interval estimation works well if

- $X_i$ has a normal distribution, for any $n > 1$
- $X_i$ has any other distribution but $n$ is large

### normal "upper-tail quantile" $z_p$

let $Z \sim N(0,1)$. for $0 < p < 1$, let $z_p$ be such that

$$p = \Pr(Z > z_p)$$

- e.g. $z_{0.5} = 0$
- $z_p = (1-p)$-quantile of $Z$
- for $0 < p < 0.5$, $\Pr(-z_p \leq Z \leq z_p) = 1 - 2p$

### (case 1) normal distribution with known $\sigma^2$

assume $X_1, \dots, X_n$ are IID $\sim N(0,1)$ with known $\sigma^2$.
for $0 < \alpha < 1$, $\Pr(-z_{\frac{\alpha}{2}} \le Z \le z_{\frac{\alpha}{2}}) = 1 - \alpha$

**confidence interval for** $\mu$: the random interval

$$\left(\bar{X} - z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}, \bar{X} + z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\right)$$

contains $\mu$ with probability $1-\alpha$,
and produces the realisation $\left(\bar{x} - z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}, \bar{x} + z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\right)$

- $1-\alpha$ is the **confidence level**
- *Proof.* since $\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$,
  - $\Pr(-z_{\frac{\alpha}{2}} \le \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \le z_{\frac{\alpha}{2}}) = 1-\alpha$
  - $\Pr(\bar{X} - z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}} \le \mu \le \bar{X} + z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}) = 1-\alpha$

### (case 2) normal distribution with unknown $\sigma^2$

assume $X_1, \dots, X_n$ are IID $\sim N(\mu, \sigma^2)$ with unknown $\sigma^2$.
replace $\sigma$ with $S$:

for $0 < p < 1$, let $t_{p,n}$ be such that
$\Pr(t_n > t_{p,n}) = p$

- $t_{p,n}$ is the *upper* $p$ quartile of the $t$ distribution with $n$ degrees of freedom
  - e.g. $t_{0.1,5} = 1.48$ (using $qt(0.9, 5)$)
- as $n \to \infty$, $t_{n,p} \to z_p$
- $\frac{\bar{X}-\mu}{S/\sqrt{n}} \sim t_{n-1}$
- $\Pr(\bar{X} - t_{\frac{\alpha}{2},n-1}\frac{S}{\sqrt{n}} \le \mu \le \bar{X} + t_{\frac{\alpha}{2},n-1}\frac{S}{\sqrt{n}})$

the random interval

$$\left(\bar{X} - t_{\frac{\alpha}{2},n-1}\frac{S}{\sqrt{n}}, \bar{X} + t_{\frac{\alpha}{2},n-1}\frac{S}{\sqrt{n}}\right)$$

contains $\mu$ with probability $1-\alpha$.

- data $x_1, \dots, x_n$ give realisations $\bar{x}$ of $\bar{X}$ and $s$ of $S$, thus the random interval gives a $(1-\alpha)$-CI for $\mu$:
  $\left(\bar{x} - t_{\frac{\alpha}{2},n-1}\frac{s}{\sqrt{n}}, \bar{x} + t_{\frac{\alpha}{2},n-1}\frac{s}{\sqrt{n}}\right)$

### (case 3) general distribution with unknown $\sigma^2$

IID $X_1, \dots, X_n$ with $E(X_i) = \mu$, $\mathrm{var}(X_i) = \sigma^2$ unknown

- for large $n$, approximately $\frac{S_n - n\mu}{\sqrt{n}\sigma} \sim N(0,1)$
- since $\frac{S_n - n\mu}{\sqrt{n}\sigma} \sim N(0,1) = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}}$ and $S \approx \sigma$ for large $n$,
  - $\Pr(-z_{\frac{\alpha}{2}} \le \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \le z_{\frac{\alpha}{2}}) \approx 1-\alpha$
  - $\Pr(\bar{X} - z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}} \le \mu \le \bar{X} + z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}) \approx 1-\alpha$

for large $n$, the random interval

$$\left(\bar{X} - z_{\frac{\alpha}{2}}\frac{S}{\sqrt{n}}, \bar{X} + z_{\frac{\alpha}{2}}\frac{S}{\sqrt{n}}\right)$$

contains $\mu$ with probability $\approx 1-\alpha$

- data $x_1, \dots, x_n$ give realisations $\bar{x}$ of $\bar{X}$ and $s$ of $S$.
- $\left(\bar{x} - z_{\frac{\alpha}{2}}SE, \bar{x} + z_{\frac{\alpha}{2}}SE\right)$
  is an *approximate* $(1-\alpha)$-CI for $\mu$.
  - $SE = \frac{s}{\sqrt{n}}$
  - for SRS, multiply $SE$ by correction factor $\sqrt{\frac{N-n}{N-1}}$
- contains $\mu$ with probability $< 1-\alpha$
  - probability $\to 1-\alpha$ as $n \to \infty$
- **exception**: for Bernoulli, $\sigma = \sqrt{p(1-p)}$ is not estimated by $s$, but by replacing $p$ with the sample proportion

# 06. METHOD OF MOMENTS

modified notation of mass/density functions:

- **bernoulli**: $f(x|p) = p^x(1-p)^{1-x}, \quad x = 0, 1$
  - parameter space is $(0,1)$
- **poisson**: $f(x|\lambda) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, \dots$
  - parameter space is $\mathbb{R}_+$

## parameter estimation

assuming data $x_1, \dots, x_n$ are realisations of IID RVs $X_1, \dots, X_n$ with mass/density function $f(x|\theta)$, where $\theta$ is unknown in parameter space $\Theta$.

- 2 methods to estimate $\theta$ :
  - method of moments (MOM)
  - method of maximum likelihood (MLE)
- for both:
  - the estimate of $\theta$ is a realisation of an estimator $\hat{\theta}$
  - SE is $SD(\hat{\theta})$
  - bias is $E(\hat{\theta}) - \theta$
- parameter space $\Theta$: set of values that can be used to estimate the real parameter value $\theta$

## Moments of an RV

the $k$-th moment of an RV $X$ is
$\mu_k = E(X^k), \quad k = 1, 2, \dots$

### estimating moments

let $X_1, \dots, X_n$ be IID with the same distribution as $X$.

the $k$-th sample moment is
$\hat{\mu}_k = \frac{1}{n}\sum_{i=1}^n X_i^k$

- $E(\hat{\mu}_k) = \mu_k \quad \Rightarrow$ unbiased estimator!
- $\hat{\mu}_k$ is an estimator of $\mu_k$. For realisations $x_1, \dots, x_n$, the realisation $\frac{1}{n}\sum_{i=1}^n x_i^k$ is an *unbiased* estimate of $\mu_k$.
- hat ( $\hat{}$ ) means estimator (random variable)
  - note that this violates the uppercase=RV, lowercase=(fixed)realisation notation
  - $\bar{x} = \frac{1}{n}\sum_{i=1}^n x_i$

### MOM: Poisson

assume $x_1, \dots, x_n$ are realisations of IID $Poisson(\lambda)$ RVs $X_1, \dots, X_n$. Let $\lambda$ be the mean number of emissions per 10 seconds ($\lambda$ is a parameter).

- let $X \sim Poisson(\lambda)$. $\mu_1 = \lambda$. Estimate $\lambda$ by estimating $\mu_1$ using sample mean $\bar{x}$, which is an estimator of $\bar{X}$.
- the MOM estimator is $\hat{\lambda} = \hat{\mu_1} = \bar{X}$
  - the random sample mean
- $\mathrm{var}(X) = \lambda$, $\mathrm{var}(\bar{X}) = \frac{\lambda}{n}$, SE = SD of estimator $= \sqrt{\frac{\lambda}{n}}$

$$\lambda \approx \bar{x} \pm \sqrt{\frac{\bar{\lambda}}{n}}$$

### MOM: Bernoulli

Assume $X_1, \dots, X_n$ are iid $Bernoulli(p)$ RVs.
Finding MOM estimator of $p$:

- let $X \sim Bernoulli(p)$. $\quad \Rightarrow \mu_1 = p$
- MOM estimator, $\hat{p} = \hat{\mu_1} = \bar{X}$
  - random sample proportion of 1's
- SE = SD of estimator $= \sqrt{\mathrm{var}(\hat{p})} = \sqrt{\frac{p(1-p)}{n}}$

### MOM: Normal

let $X_1, \dots, X_n$ be iid $N(\mu, \sigma^2)$ with parameters $\mu, \sigma^2$
for $X \sim N(\mu, \sigma^2)$: parameter space, $\Theta = \mathbb{R} \times \mathbb{R}_+$

1. $\mu_1 = \mu, \quad \mu_2 = \sigma^2 + \mu^2$
2. express $\mu = \mu_1$; $\sigma^2 = \mu_2 - \mu_1^2$; then add hats
3. MOM estimators:

$$\hat{\mu} = \bar{X} \qquad \hat{\sigma^2} = \frac{1}{n}\sum_{i=1}^n (X_i - \bar{X})^2$$

(to construct CI for $\sigma^2$: use $S^2 \quad \Rightarrow$ since $E(S^2) = \sigma^2$)

### MOM: Geometric

let $x_1, \dots, x_n$ be realisations of IID $Geometric(p)$ RVs $X_1, \dots, X_n$ with expectation $1/p$.

- for $X \sim Geometric(p) \quad \Rightarrow E(X) = \frac{1}{p}$
  - $\Pr(X = i) = p(1-p)^{i-1}$ for $i = 1, 2, \dots$
  - $E(X) = \sum_{i=1}^\infty ip(1-p)^{i-1} = \frac{1}{p}$
- $\mu_1 = \frac{1}{p} \quad \Rightarrow p = \frac{1}{\mu_1} \quad \Rightarrow \hat{p} = \frac{1}{\bar{X}}$
- MOM estimator, $\hat{p} = \frac{1}{\bar{X}}$
  - then MOM estimate $= \frac{1}{\bar{x}}$
- SE = $SD(1/\bar{X}) \quad \Rightarrow$ use monte carlo to approximate

### MOM: Gamma

let $X_1, \dots, X_n$ be iid $Gamma(\alpha, \lambda)$ RVs with shape parameter $\alpha > 0$, rate parameter $\lambda > 0$

- $X \sim Gamma(\alpha, \lambda)$, $E(X) = \frac{\alpha}{\lambda}$, $E(X^2) = \frac{\alpha(\alpha+1)}{\lambda^2}$
- express parameters in terms of moments:
  $\mu_1 = \frac{\alpha}{\lambda}$, $\mu_2 - \mu_1^2 = \frac{\alpha}{\lambda^2} \quad \Rightarrow \lambda = \frac{\mu_1}{\mu_2 - \mu_1^2}, \alpha = \lambda\mu_1$
- MOM estimators: $\hat{\alpha} = \frac{\bar{X}^2}{\hat{\sigma}^2}$, $\hat{\lambda} = \frac{\bar{X}}{\hat{\sigma}^2}$

### MOM estimators are consistent

let $X_1, \dots, X_n$ be iid with mass/density $f(x|\theta)$, where $\theta \in \Theta \subset \mathbb{R}$.
Suppose $\theta = g(\mu_1)$ for some *continuous* function $g$.
Then the MOM estimator is **consistent** (approaches $\theta$ with more data)

- the MOM estimator is $\hat{\theta} = g(\hat{\mu}_1)$. as $n \to \infty$, $\hat{\mu}_1 \to \mu_1$
- since $g$ is continuous, $\quad \hat{\theta} \to g(\mu_1) = \theta$
  - **asymptotic unbiasedness**: $E(\hat{\theta}) \to \theta$

# 07. MLE

MOM: works through estimating moments - if no formula is available for $SD(\hat{\theta})$ or $E(\hat{\theta})$, monte carlo can be used
MLE: another estimation method

## Likelihood function

let $x_1, \dots, x_n$ be realisations of iid rvs $X_1, \dots, X_n$ with density $f(x|\theta)$, $\theta \in \Theta \subset \mathbb{R}^k$.

**likelihood function** $L : \Theta \to \mathbb{R}_+$ is
$L(\theta) = f(x_1|\theta) \times \cdots \times f(x_n|\theta) = \prod_{i=1}^n f(x_i|\theta)$
**loglikelihood function** $\ell : \Theta \to \mathbb{R}$ is
$\ell(\theta) = \log L(\theta) = \sum_{i=1}^n \log f(x_i|\theta)$

## Maximum Likelihood Estimation (MLE)

$$L(\theta) = \prod_{i=1}^n f(x_i|\theta)$$

- **maximiser** of $L \to$ the maximum likelihood estimate of $\theta$ (a realisation of the MLEstimator $\hat{\theta}$)
  - maximiser of loglikelihood $\ell = \log L$ over $\Theta$

### poisson (log)likelihood/MLE

$Poisson(\lambda) : f(x|\lambda) = \frac{\lambda^x e^{-\lambda}}{x!}, \; x = 0, 1, 2, \dots$

- let $x_1, \dots, x_n$ be realisations of iid Poisson($\lambda$) RVs $X_1, \dots, X_n$. the joint probability of data is
  $f(x_1|\lambda) \times \cdots \times f(x_n|\lambda) = \frac{\lambda^{\sum_{i=1}^n x_i} e^{-n\lambda}}{x_1! \dots x_n!}$
- **likelihood**: probability as a function of only $\lambda$
  $L(\lambda) = \frac{\lambda^{\sum_{i=1}^n x_i} e^{-n\lambda}}{x_1! \dots x_n!}$
  - we can leave out constant factors:
    $L(\lambda) = \lambda^{\sum_{i=1}^n x_i} e^{n\lambda}$
- **loglikelihood:**
  $\ell(\lambda) = (\sum_{i=1}^n x_i) \log\lambda - n\lambda - \sum_{i=1}^n \log(x_i!)$
  - leaving out additive constants:
    $\ell(\lambda) = (\sum_{i=1}^n x_i) \log\lambda - n\lambda$
- **MLE of** $\lambda = \bar{x}$ (maximiser of $L(\lambda)$)
  - differentiate $\ell(\lambda)$: $\ell'(\lambda) = \frac{\sum_{i=1}^n x_i}{\lambda} - n$
  - $\ell'(\lambda) = 0 \quad \Rightarrow \lambda = \bar{x}$
  - $\ell''(\lambda) < 0$ (thus max point)

### normal (log)likelihood/MLE

$N(\mu, \sigma^2)$: for $x \in \mathbb{R}$,
$f(x|\mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} = (2\pi)^{\frac{1}{2}}\sigma^{-1}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

- let $x_1, \dots, x_n$ be realisations of iid N($\mu, \sigma$) RVs $X_1, \dots, X_n$. the joint probability of data is
  $f(x_1|\lambda) \times \cdots \times f(x_n|\lambda) = \frac{\lambda^{\sum_{i=1}^n x_i} e^{-n\lambda}}{x_1! \dots x_n!}$
- **likelihood** function: joint density as a function of $(\mu, \sigma)$

$$L(\mu, \sigma) = f(x_1|\mu, \sigma) \times \cdots \times f(x_n|\mu, \sigma)$$
$$= (2\pi)^{-\frac{n}{2}}\sigma^{-n}e^{-\frac{1}{2\sigma^2}\sum_{i=1}^n (x_i - \mu)^2}$$

- **loglikelihood:**
  $\ell(\mu, \sigma) = -\frac{n}{2}\log(2\pi) - n\log\sigma - \frac{1}{2\sigma^2}\sum_{i=1}^n (x_i - \mu)^2$
- **MLE:**
  - MLE of $\mu = \bar{x}$
  - MLE of $\sigma = \hat{\sigma} = \sqrt{\frac{1}{n}\sum_{i=1}^n (x_i - \bar{x})^2}$

### Gamma distribution

$Gamma(\alpha, \lambda) : \; f(x|\alpha, \lambda) = \frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\lambda x}, \; x > 0$

- **log of density:** $\alpha\log\lambda - \log\Gamma(\alpha) + (\alpha - 1)\log x - \lambda x$
- **loglikelihood:**

$$n\alpha\log\lambda - n\log\Gamma(\alpha) + (\alpha-1)\sum_{i=1}^n \log x_i - \lambda\sum_{i=1}^n x_i$$

  - if $\alpha$ is known, then $\ell(\lambda) = n\alpha\log\lambda - \lambda\sum_{i=1}^n x_i$
- differentiate $\Rightarrow$ the ML estimates of $(\alpha, \lambda)$ satisfy
  $\log(\frac{\alpha}{\bar{x}}) - \frac{\Gamma'(\alpha)}{\Gamma(\alpha)} + \bar{y} = 0, \quad \lambda = \frac{\alpha}{\bar{x}}$ where
  $\bar{y} = \frac{1}{n}\sum_{i=1}^n \log x_i$
- the **ML estimators** $(\hat{\alpha}, \hat{\lambda})$ satisfy
  $\log(\frac{\alpha}{\bar{X}}) - \frac{\Gamma'(\alpha)}{\Gamma(\alpha)} + \bar{Y} = 0, \quad \lambda = \frac{\alpha}{\bar{X}}$
  - $\log(\frac{\hat{\alpha}}{\bar{X}}) - \frac{\Gamma'(\hat{\alpha})}{\Gamma(\hat{\alpha})} + \bar{Y} = 0, \quad \hat{\lambda} = \frac{\hat{\alpha}}{\bar{X}}$

## ML vs MOM

- MOM estimates can always be written in terms of the data (sample moments)
  - ML uses *
- ML has better (smaller) SE and bias than MOM
- ML estimates are functions of $\bar{x}$ and $\bar{y}$. MOM never uses $\bar{y}$

## Kullback-Liebler divergence (KL)

let $\mathbf{q} = (q_1, \ldots, q_k)$ and $\mathbf{p} = (p_1, \ldots, p_k)$ be strictly positive probability vectors.

the **KL divergence** between $\mathbf{q}$ and $\mathbf{p}$ is

$$d_{KL}(\mathbf{q}, \mathbf{p}) = \sum_{i=1}^{k} q_i \log(\frac{q_i}{p_i})$$

- $d_{KL}(\mathbf{q}, \mathbf{p}) \geq 0$ (equality $\iff$ $\mathbf{q} = \mathbf{p}$)
- $d_{KL}(\mathbf{q}, \mathbf{p}) \neq d_{KL}(\mathbf{p}, \mathbf{q})$

### Multinomial

let $(x_1, \ldots, x_n)$ be strictly positive realisations from $(X_1, \ldots, X_n) \sim Multinomial(n, \mathbf{p})$.

- $L(\mathbf{p}) = \Pr(X_1 = x_1, \ldots, X_k = x_k) = cp_1^{x_1} \ldots p_k^{x_k}$
  $= p_1^{x_1} \ldots p_k^{x_k}$ (simplified)
- $\ell(\mathbf{p}) = x_1 \log p_1 + \cdots + x_k \log p_k$
- maximising $\ell$ via KL divergence
  - if $x$ is from $X \sim Binom(n, p)$, the MOM and ML estimates are both $\hat{p} = \frac{x}{n}$
    - the MOM estimate of $p_i$ is $q_i = \frac{x_i}{n}$.
  - for any $\mathbf{p}$,
    $\ell(\mathbf{q}) - \ell(\mathbf{p}) = \sum_{i=1}^{k} x_i \log q_i - \sum_{i=1}^{k} x_i \log p_i$
    $= n\, d_{KL}(\mathbf{q}, \mathbf{p}) \geq 0$
    - $\ell(\mathbf{q}) - \ell(\mathbf{p}) = 0 \iff \mathbf{p} = \mathbf{q}$

## Hardy-Weinberg equilibrium (HWE)

let $\theta$ be the proportion of $a$.

the population is in **HWE** if

$$f(aa) = \theta^2, \quad f(aA) = 2\theta(1-\theta), \quad f(AA) = (1-\theta)^2$$

- (e.g. genotypes) Under HWE, the number of $a$ alleles in an individual has a $Binom(2, \theta)$ distribution
  - for $n$ randomly chosen people, number of $a$ alleles $(AA, Aa, aa) \sim Multinomial(n, \theta)$

### Multinomial ML estimation

for $(X_1, X_2, X_3) \sim Multinomial(n, \mathbf{p})$
where $p_1 = (1-\theta)^2$, $p_2 = 2\theta(1-\theta)$, $p_3 = \theta^2$

- $L(\theta) = (1-\theta)^{2x_1} 2^{x_2} \theta^{x_2} (1-\theta)^{x_2} \theta^{2x_3}$
  $= 2^{x_2}\, (1-\theta)^{2x_1+x_2}\, \theta^{x_2+2x_3}$
- $\ell(\theta) = x_2 \log 2 + (2x_1 + x_2)\log(1-\theta) + (x_2 + 2x_3)\log\theta$
- ML estimator: $\hat{\theta} = \frac{X_2 + 2X_3}{2n}$
- SE estimation: $\sqrt{\frac{\theta(1-\theta)}{2n}}$
  - $X_2 + 2X_3$ is the number of $a$ alleles: $Binom(2n, \theta)$
    $\Rightarrow \text{var}(\hat{\theta}) = \frac{\theta(1-\theta)}{2n}$

# 08. LARGE-SAMPLE DISTRIBUTION OF MLEs

let $X_1, \ldots, X_n$ be iid $Geometric(0.5)$ RVs, with mean $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$.
by CLT, $\bar{X}_n$ and $\frac{1}{\bar{X}_n}$ have a normal distribution.

### asymptotic normality of ML estimator

let $\hat{\theta}_n$ be the ML estimator of $\theta \in \Theta \subset \mathbb{R}$, based on iid RVs $X_1, \ldots, X_n$ with density $f(x|\theta)$.

for large $n$, the distribution of $\hat{\theta}_n$ is approximately

$$N(\theta, \frac{\mathcal{I}(\theta)^{-1}}{n})$$

where $\mathcal{I}(\theta)$ is the Fisher information derived from $f(x|\theta)$

- $\hat{\theta}_n$ is asymptotically unbiased (like MOM)
  - $E(\hat{\theta}_n) \neq \theta$ (biased)

## Fisher Information

let $X$ have density $f(x|\theta)$, $\theta \in \Theta \subset \mathbb{R}^p$.

the **Fisher information** is the $p \times p$ matrix

$$\mathcal{I}(\theta) = -E\left[\frac{d^2 \log f(X|\theta)}{d\theta^2}\right]$$

- $\mathcal{I}(\theta)$ is symmetric, with $(ij)$-entry $-E\left[\frac{\delta^2 \log f(X|\theta)}{\delta\theta_i \delta\theta_j}\right]$
- $\mathcal{I}(\theta)$ measures the information about $\theta$ in one sample $X$.

### Asymptotic normality: Bernoulli

$X \sim Bernoulli(p): \; f(x|p) = p^x(1-p)^{1-x}, \; x = 0, 1$

**Fisher information**

- $\log f(X|p) = X \log p + (1-X)\log(1-p)$
- differentiate $\frac{d}{dp}$: $\frac{X}{p} - \frac{1-X}{1-p}$
- differentiate $\frac{d^2}{dp^2}$: $-\frac{X}{p^2} - \frac{1-X}{(1-p)^2}$
- $\mathcal{I}(p) = -E(\frac{d^2 \log f(X|p)}{dp^2}) = \frac{1}{p(1-p)}$
  - minimised at $p = 0.5$

**Asymptotic normality**

for $X_1, \ldots, X_n$ iid $Bernoulli(p)$ RVs,
Fisher information in each $X_i$: $\mathcal{I}(p) = \frac{1}{p(1-p)}$

- ML estimator $\hat{p} = \bar{X}$
- for large $n$, $\hat{p} \approx N\left(p, \frac{p(1-p)}{n}\right)$
  - $E(\hat{p}) = p$, $\quad \text{var}(\hat{p}) = \frac{p(1-p)}{n}$

### Asymptotic normality: Geometric

$X \sim Geometric(p): \; f(x|p) = p(1-p)^{1-x}$

**Fisher information**

- $\log f(X|p) = \log p + (X-1)\log(1-p)$
- differentiate $\frac{d}{dp}$: $\frac{1}{p} - \frac{X-1}{1-p}$
- differentiate $\frac{d^2}{dp^2}$: $-\frac{1}{p^2} - \frac{X-1}{(1-p)^2}$
- $\mathcal{I}(p) = -E(\frac{d^2 \log f(X|p)}{dp^2}) = \frac{1}{p(1-p)} + \frac{1}{p^2} = \frac{1}{p^2(1-p)}$

**Asymptotic normality**

for $X_1, \ldots, X_n$ iid $Geometric(p)$ RVs,
Fisher information in each $X_i$, $\mathcal{I}(p) = \frac{1}{p^2(1-p)}$

- ML estimator $\hat{p} = \frac{1}{\bar{X}}$
- for large $n$, distribution of $\hat{p} \approx N\left(p, \frac{p^2(1-p)}{n}\right)$
  - $E(\hat{p}) > p$ since $E(\hat{p}) = E(\frac{1}{\bar{X}}) > \frac{1}{E(\bar{X})} = p$
  - likely $\text{var}(\hat{p}) \neq \frac{p^2(1-p)}{n}$

### Asymptotic normality: Normal

**Fisher information**

$X \sim N(\mu, \sigma^2), \theta = (\mu, \sigma)$.

$f(x|\theta) = \frac{1}{\sqrt{2\pi}\sigma}\exp\{-\frac{(x-\mu)^2}{2\sigma^2}\}, \quad x \in \mathbb{R}$

- $\log f(X|\theta) = \frac{1}{2}\log 2\pi - \log\sigma - \frac{(x-\mu)^2}{2\sigma^2 n}$
  $= c - \log\sigma - \frac{(X-\mu)^2}{2\sigma^2 n}$
- differentiate $\frac{d}{d\theta}$: $\frac{\delta}{\delta\mu} = \frac{X-\mu}{\sigma^2}$, $\quad \frac{\delta}{\delta\sigma} = -\frac{1}{\sigma} + \frac{(X-\mu)^2}{\sigma^3}$
- differentiate $\frac{d^2}{d\theta^2}$: $\begin{bmatrix} \frac{\delta^2}{\delta\mu^2} & \frac{\delta^2}{\delta\mu\delta\sigma} \\ \frac{\delta^2}{\delta\sigma\delta\mu} & \frac{\delta^2}{\delta\sigma^2} \end{bmatrix}$

- $\mathcal{I}(\theta) = -E(\frac{d^2 \log f(X|\theta)}{d\theta^2}) = \begin{bmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & \frac{2}{\sigma^2} \end{bmatrix}$

**Asymptotic normality**

for $X_1, \ldots, X_n$ iid $N(\mu, \sigma^2)$ RVs, $\theta = (\mu, \sigma)$,

Fisher information in each $X_i : \mathcal{I}(\theta) = \begin{bmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & \frac{2}{\sigma^2} \end{bmatrix}$

- ML estimator $\hat{\theta} = \begin{bmatrix} \bar{X} \\ \hat{\sigma} \end{bmatrix}$
- for large $n$, $\hat{\theta} \approx N\left(\begin{bmatrix} \mu \\ \sigma \end{bmatrix}, \begin{bmatrix} \frac{\sigma^2}{n} & 0 \\ 0 & \frac{\sigma^2}{2n} \end{bmatrix}\right)$

**are expectation and variance exact?**

- a positive random variable cannot be exactly normal!
  - $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$ $\Rightarrow$ exact
  - $\hat{\sigma} \sim N(\sigma, \frac{\sigma^2}{2n})$ $\Rightarrow$ not exact! $E(\hat{\sigma}) \neq \sigma$ since $\sigma \geq 0$

**normal data**

for $x_1, \ldots, x_n$ IID $N(\mu, \sigma^2)$ RVs with large $n$,
ML estimates of $\mu$ and $\sigma$ are $\bar{x} = \ldots$ and $\hat{\sigma} = \ldots$

- for approximate variance $\begin{bmatrix} \frac{\sigma^2}{n} & 0 \\ 0 & \frac{\sigma^2}{2n} \end{bmatrix}$,
  SEs of $\bar{x}$ and $\hat{\sigma}$ are estimated as $\frac{\hat{\sigma}}{\sqrt{n}}$ and $\frac{\hat{\sigma}}{\sqrt{2n}}$
- approximate $(1-\alpha)$-CI:
  $\mu : \left(\bar{x} - z_{\frac{\alpha}{2}}\frac{\hat{\sigma}}{\sqrt{n}}, \bar{x} + z_{\frac{\alpha}{2}}\frac{\hat{\sigma}}{\sqrt{n}}\right)$
  $\sigma : \left(\hat{\sigma} - z_{\frac{\alpha}{2}}\frac{\hat{\sigma}}{\sqrt{2n}}, \hat{\sigma} + z_{\frac{\alpha}{2}}\frac{\hat{\sigma}}{\sqrt{2n}}\right)$

## Gamma distribution

$X \sim Gamma(\alpha, \lambda)$,
$f(x|\alpha, \lambda) = \frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\lambda x}, \; x > 0$
$\log f(X) = \alpha\log\lambda - \log\Gamma(\alpha) + (\alpha-1)\log X - \lambda X$
let $\psi(\alpha) = \frac{d}{d\alpha}\log\Gamma(\alpha)$:
($\psi(\alpha) =$ digamma function, $\psi'(\alpha) =$ trigamma function)

- $\frac{\delta \log f(X)}{\delta\alpha} = \log\lambda - \psi(\alpha) + \log X$
- $\frac{\delta \log f(X)}{\delta\lambda} = \frac{\alpha}{\lambda} - X$
- $\frac{\delta^2 \log f(X)}{\delta\alpha^2} = -\psi'(\alpha)$
- $\frac{\delta^2 \log f(X)}{\delta\lambda^2} = -\frac{\alpha}{\lambda^2}$
- $\frac{\delta^2 \log f(X)}{\delta\alpha\delta\lambda} = \frac{\delta^2 \log f(X)}{\delta\lambda\delta\alpha} = \frac{1}{\lambda}$

$$\mathcal{I}(\alpha, \lambda) = \begin{bmatrix} \psi'(\alpha) & -\frac{1}{\lambda} \\ -\frac{1}{\lambda} & \frac{\alpha}{\lambda^2} \end{bmatrix}$$

## Approximate CI with ML estimate

$\hat{\theta}_n$ is the ML estimator of $\theta \in \Theta \subset \mathbb{R}$ based on iid RVs $X_1, \ldots, X_n$. $0 < \alpha < 1$

- for large $n$, approximately $\hat{\theta}_n \sim N(\theta, \frac{\mathcal{I}(\theta)^{-1}}{n})$.
  for $0 < \alpha < 1$,
  $1 - \alpha \approx \Pr\left(-z_{\frac{\alpha}{2}} \leq \frac{\hat{\theta}_n - \theta}{\sqrt{\mathcal{I}(\theta)^{-1}/n}} \leq z_{\frac{\alpha}{2}}\right)$
- the random interval
  $\left(\hat{\theta}_n - z_{\frac{\alpha}{2}}\sqrt{\frac{\mathcal{I}(\theta)^{-1}}{n}}, \hat{\theta}_n + z_{\frac{\alpha}{2}}\sqrt{\frac{\mathcal{I}(\theta)^{-1}}{n}}\right)$
  covers $\theta$ with probability $\approx 1 - \alpha$
- **MLE**: ML estimate of $\theta$, **SE**: $\sqrt{\frac{\mathcal{I}(\theta)^{-1}}{n}}$ with $\theta$ replaced by MLE
  - approxiate $(1-\alpha) - CI$ for $\theta$ is
    $(MLE - z_{\frac{\alpha}{2}}SE, MLE + z_{\frac{\alpha}{2}}SE)$

## Scope of asymptotic normality of ML estimators

- for iid normal RVs, let $\hat{\sigma}$ be the ML estimator of $\sigma$. then $\hat{\sigma}^2$ is the ML estimator of $\sigma^2$
  - both $\hat{\sigma}$ and $\hat{\sigma}^2$ are asymptotically normal
  - $\frac{1}{\hat{\sigma}}$ is also asymptotically normal
- let $\hat{\theta}^n$ be the ML estimator of $\theta$. For strictly increasing or strictly decreasing $h : \Theta \to \mathbb{R}$, $h(\hat{\theta}^n)$ is the ML estimator of $h(\theta)$.
  - for large $n$, $h(\hat{\theta}^n)$ is approximately normal

## population mean vs parameter

for $n$ random draws with replacement from a population with mean $\mu$ and variance $\sigma^2$,

| Estimator | $E$ | var | Distribution |
|---|---|---|---|
| random sample mean, $\hat{\mu}$ | $\mu$ | $\frac{\sigma^2}{n}$ | $\approx$ normal |
| ML estimator, $\hat{\theta}_n$ | $\approx \theta$ | $\approx \frac{\mathcal{I}(\theta)^{-1}}{n}$ | $\approx$ normal |

$\hat{\theta}_n$ is not normal (but may approach normal for large $n$)

## summary

let $X$ have density $f(x|\theta), \theta \in \Theta \subset \mathbb{R}^k$.
The **Fisher information** at $\theta$ in $X$ is the $k \times k$ matrix
$-E\left[\frac{d^2 \log f(X|\theta)}{d\theta^2}\right]$.

let $\hat{\theta}_n$ be the ML estimator of $\theta$ based on iid RVs $X_1, \ldots, X_n$ with density $f(x|\theta)$.
For large $n$, the distribution of $\hat{\theta}_n$ is approximately
$N\left(\theta, \frac{\mathcal{I}(\theta)^{-1}}{n}\right)$

$\Rightarrow$ SE can be estimated without monte carlo
$\Rightarrow$ accurate CIs are available

skipped: Fisher information in IID samples; binomial fisher information, MLE; HWE trinomial fisher information
$E(\frac{d \log f(X|\lambda)}{d\lambda}) = 0$

## Cramér-Rao inequality

if $\hat{\theta}_n$ is unbiased, then $\text{var}(\hat{\theta}_n) \geq \frac{\mathcal{I}(\theta)^{-1}}{n}$
**efficient** $\iff$ equality

# 09. HYPOTHESIS TESTING

let $x_1, \ldots, x_n$ be realisations of IID $N(\mu, \sigma^2)$ RVs $X_1, \ldots, X_n$ where $\mu$ is a parameter and $\sigma$ is known.

**null hypothesis**, $H_0 : \mu = \mu_0$
**alternative hypothesis**, $H_1 : \mu = \mu_1$

It is believed that $\mu = \mu_0$, but it might be $\mu_1$. 2 methods to test if $H_0$ should be rejected in favour of $H_1$ using $\bar{x}$:

- if $\bar{x}$ falls inside the **rejection region**, we reject $H_0$
  - based on a choice of $\alpha$ (type $I$ error)
- ***P* value** $\to$ the probability that $\bar{X}$ is more extreme than $\bar{x}$, assuming $H_0$ is true. (if small, doubt $H_0$)
  - based on an observed test statistic

if $\sigma$ is unknown or $x_1, \ldots, x_n \not\sim N(\mu, \sigma^2)$, we can use CLT

## Rejection region

$x_1, \ldots, x_n$ are from IID $N(\mu, \sigma^2)$ RVs, with $\sigma$ known

## One-tailed test

$$H_0 : \mu = \mu_0, \quad H_1 : \mu = \mu_1 > \mu_0$$

under $H_0$,

$$\bar{X} \sim N(\mu_0, \frac{\sigma^2}{n}), \quad \frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}} \sim N(0,1)$$

$$\alpha = P_{H_0}(\bar{X} > \mu_0 + c) = \Pr(Z > \frac{c}{\sigma/\sqrt{n}}) \quad \Rightarrow c = z_\alpha \frac{\sigma}{\sqrt{n}}$$

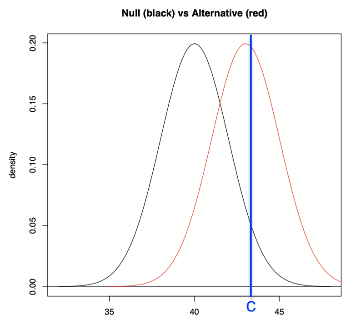
- reject $H_0$ if $\bar{x} - \mu_0 > c$ (for some $c > 0$)
  - $\bar{x}$ is the **test statistic**
  - interval $(\mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}}, \infty)$ is the **rejection region**
    - for a test of **size** $\alpha$, $c = z_\alpha \frac{\sigma}{\sqrt{n}}$

| Hypothesis | $\bar{x} < \mu_0 + c$ | $\bar{x} > \mu_0 + c$ |
|---|---|---|
| $H_0 : \mu = \mu_0$ | ✓ not reject $H_0$ | ×$(I)$ reject $H_0$ |
| $H_1 : \mu = \mu_1$ | ×$(II)$ not reject $H_0$ | ✓ reject $H_0$ |

- type $I$ error: rejecting $H_0$ when it is true
- type $II$ error: not rejecting $H_0$ when it is false

## Size and power

- **size** of a test → probability of a Type $I$ error
  - $\alpha := P_{H_0}(\bar{X} > \mu_0 + c)$
  - aka **level**
- **power** of a test → 1− probability of a Type $II$ error
  - $\beta := P_{H_1}(\bar{X} > \mu_0 + c) \quad \Rightarrow$ power $= 1 - \beta$
  - as $n \to \infty$, power $\to 1$
    - increasing power of rejecting $H_0$
- $\alpha$ and $\beta$ are both about the same event ($\bar{X}$ is in the rejection region), but calculated under different hypotheses $(H_0, H_1)$
- $\uparrow c : \ \downarrow \alpha, \downarrow \beta$ ($\downarrow$ type $I$ error, $\uparrow$ type $II$ error)
- commonly $\alpha = 0.05$
  - keep $\alpha$ small since $H_0$ is the default hypothesis




## Two-tailed test

$x_1, \dots, x_n$ are from iid $N(\mu, \sigma^2)$ RVs, $\sigma$ known

$$H_0 : \mu = \mu_0, \quad H_1 : \mu = \mu_1 \neq \mu_0$$

- reject $H_0$ if $|\bar{x} - \mu_0| > c$, for some $c > 0$
  - **rejection region:** $(-\infty, \mu_0 - c)$ and $(\mu_0 + c, \infty)$
- $\alpha = P_{H_0}(|\bar{X} - \mu_0| > c) = \Pr\left(|Z| > \frac{c}{\sigma/\sqrt{n}}\right)$
  $= 2\Pr\left(Z > \frac{c}{\sigma/\sqrt{n}}\right)$
- $c = z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$
  - **rejection region:** $(-\infty, \mu_0 - z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}) \wedge (\mu_0 + z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}, \infty)$

## Composite hypothesis

- **simple** hypothesis → specify a single value ($H_0 : \mu = \mu_0, H_1 : \mu = \mu_1$)
- **composite** hypothesis → range of values
  - one-tailed test: $H_0 : \mu = \mu_0, \ H_1 : \mu > \mu_0$
    - rejection region: $(\mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}}, \infty)$
    - $\Rightarrow$ no change since it doesn't involve $\mu_1$
  - two-tailed test: $H_0 : \mu = \mu_0, \ H_1 : \mu \neq \mu_0$
    - rejection region: $(-\infty, \mu_0 - z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}) \wedge (\mu_0 + z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}, \infty)$
    - $\Rightarrow$ no change since it doesn't involve $\mu_1$
    - if $\bar{x}$ falls *outside* the rejection region, i.e. $\mu_0 - z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} \leq \bar{x} \leq \mu_0 + z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$
      - then $H_0$ is NOT rejected at level $\alpha$
      - $\mu_0$ lies in the $(1-\alpha)$-CI for $\mu$
  - as $n \to \infty$, power $\to 1$

## Hypothesis testing and CI

the $(1-\alpha)$-CI for $\mu$, $\left(\bar{x} - z_{\frac{\alpha}{2}} \frac{\hat{\sigma}}{\sqrt{n}}, \bar{x} + z_{\frac{\alpha}{2}} \frac{\hat{\sigma}}{\sqrt{n}}\right)$
consists of the values $\mu_0$ for which the test
$H_0 : \mu = \mu_0, \ H_1 : \mu \neq \mu_0$ is not rejected at level $\alpha$.

## $P$-value

- **$P$-value** → the probability under $H_0$ that the random test statistic is more extreme than the observed test statistic
  - small $p$-value = more "extreme" (more doubt)
- reject $H_0$ at level $\alpha \iff P < \alpha$
- generally, $P$-value for two-tailed test is double that of one-tailed test

### formulae for $P$-value

$H_1 : \mu > \mu_0$

$$P = P_{H_0}(\bar{X} > \bar{x}) = \Pr\left(Z > \frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}\right)$$

$H_1 : \mu < \mu_0$

$$P = P_{H_0}(\bar{X} < \bar{x}) = \Pr\left(Z < \frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}\right)$$

$H_1 : \mu \neq \mu_0$

$$P = P_{H_0}(|\bar{X} - \mu_0| > |\bar{x} - \mu_0|) = \Pr\left(|Z| > \frac{|\bar{x}-\mu_0|}{\sigma/\sqrt{n}}\right)$$

# 10. GOODNESS-OF-FIT

- **likelihood ratio** (LR) test → based on the ratio of likelihoods
  - $P$-value can be approximated using $\chi^2$ distribution for a large sample size

## multinomial

let $X \sim Trinomial(n, \mathbf{p})$. by HWE, $\mathbf{p}$ is a function of $\theta$ as follows: $p_1 = (1-\theta)^2, \ p_2 = 2\theta(1-\theta), \ p_3 = \theta^2$
let $L_1$ and $L_0$ be the maximum likelihood value for the general model ($Trinomial(n, \mathbf{p})$) and the HWE.

- $L_1 \geq L_0$ ($L_0$ is the maximum over a subset of $L_1$)
  - general trinomial
    - likelihood, $L(\mathbf{p}) = p_1^{x_1} p_2^{x_2} p_3^{x_3}$
    - ML estimate of $\mathbf{p}$ is $\frac{x}{n}$
    - $\log L_1 = x_1 \log(\frac{x_1}{n}) + x_2 \log(\frac{x_2}{n}) + x_3 \log(\frac{x_3}{n})$
  - HWE:
    - likelihood, $L(\theta) = p_1(\theta)^{x_1} p_2(\theta)^{x_2} p_3(\theta)^{x_3}$
    - ML estimate of $\theta$ is $\frac{x_2 + 2x_3}{2n}$
- larger $L_1/L_0 \Rightarrow$ poorer fit for HWE

## LR test

- null hypothesis: HWE holds
  $H_0 : p_1 = (1-\theta)^2, \ p_2 = 2\theta(1-\theta), \ p_3 = \theta^2$
- LR test statistic: $2\log\left(\frac{L_1}{L_0}\right) = 2(\log L_1 - \log L_0)$
- degree of freedom = difference in the number of parameters between the models
  - general model has 2 params, HWE has 1 param
- $P$-value $= \Pr\left(\chi_1^2 > 2\log(\frac{L_1}{L_0})\right)$

## Nested models

the set of all $Trinomial(n, \mathbf{p})$ distributions can be represented by

$$\Omega_1 = \left\{(p_1, p_2, p_3) : p_i > 0, \ \sum_{i=1}^3 p_i = 1\right\}$$

which has dimension 2 $\quad (\dim \Omega_1 = 2)$

- by HWE, $\mathbf{p}$ is in the subset
  $\Omega_0 = \left\{((1-\theta)^2, 2\theta(1-\theta), \theta^2) : 0 < \theta < 1\right\}$
  $(\dim \Omega_0 = 1)$
- $\Omega_0$ is **nested** in $\Omega_1$
- measure goodness-of-fit of HWE by testing $H_0 : \mathbf{p} \in \Omega_0$

## General Multinomial LR test

let $(X_1, \dots, X_k) \sim Multinomial(n, \mathbf{p})$. then $\mathbf{p} \in \Omega_1$, the set of all positive probability vectors of length $k$.

to test if $\mathbf{p}$ is in a subspace
$\Omega_0 = \left\{(p_1(\theta), \dots, p_k(\theta)) : \theta \in \Theta \subset \mathbb{R}^h\right\}$
with $\dim \Omega_0 < \dim \Omega_1 = k - 1$

let $L_j$ be the maximum likelihood value under $\Omega_j$.
To test $H_0 : \mathbf{p} \in \Omega_0$, we use the **LR statistic**,

$$G = 2\log(\frac{L_1}{L_0})$$

- for $\Omega_1$: $\log L_1 = \sum_{i=1}^k X_i \log(\frac{X_i}{n})$
- for $\Omega_0$: $\log L_0 = \sum_{i=1}^k X_i \log p_i(\hat{\theta})$

$$G = 2\sum_{i=1}^k X_i \log\left(\frac{X_i}{np_i(\hat{\theta})}\right)$$

given data $(x_1, \dots, x_n)$, let $g$ be a realisation of $G$.
$P$-value $P_{H_0}(G > g)$ is approximately
$\Pr(\chi^2_{k-1-\dim \Omega_0} > g)$ for large $n$.

- to compute $g$, replace
  - $X_i$ with *observed count* $x_i$
  - $np_i(\hat{\theta})$ with *expected count*, calculated using ML estimate of $\theta$

## Test of independence

for a population with attributes $q$ and $r$, let $p_{ij}$ be the population proportion of people with $q = q_i$ and $r = r_j$. for any $i, j$, $p_{ij} = q_i \times r_j$.

- let $(X_{ij}, 1 \leq i \leq I, 1 \leq j \leq J) \sim Multinomial(n, \mathbf{p})$.
  $\mathbf{p} \in \Omega_1$, where $\dim \Omega_1 = IJ - 1 = k - 1$.
- $H_0$ : the two categories $q, r$ are independent
  - if $q, r$ are independent, then $\exists$ positive numbers $\sum_{i=1}^I q_i = \sum_{j=1}^J r_j = 1$ such that $p_{ij} = q_i \times r_j$, $1 \leq i \leq I, 1 \leq j \leq J$
- $\dim \Omega_0 = (I-1) + (J-1) = I + J - 2$
- $\dim \Omega_1 - \dim \Omega_0 = (I-1)(J-1)$
- under independence ($H_0$), for large $n$, approximately $G \sim \chi^2_{(I-1)(J-1)}$

## G statistic

for any $i$, let $X_{i+} = \sum_{j=1}^J X_{ij}$.
for any $j$, let $X_{+j} = \sum_{i=1}^I X_{ij}$.

- $\Omega_1$ : $\log L_1 = \sum_{ij} X_{ij} \log\left(\frac{X_{ij}}{n}\right)$
- $\Omega_0$ :
  $\log L_0 = \sum_i X_{i+} \log\left(\frac{X_{i+}}{n}\right) + \sum_j X_{+j} \log\left(\frac{X_{+j}}{n}\right)$
- $G = 2(\log L_1 - \log L_0) = 2\sum_{ij} X_{ij} \log\left(\frac{X_{ij}}{X_{i+}X_{+j}/n}\right)$
- the data $x_{ij}$ are the *observed counts*
- the data $x_{i+}x_{+j}/n$ are the *expected counts*
- $P$-value $= \Pr\left(\chi^2_{(I-1)(J-1)} > g\right)$

## General LR test

we have $n$ iid RVs with density defined by $\theta \in \Omega_1$ of dimension $k_1$; nested in $\Omega_1$ is a smaller model $\Omega_0$ of dimension $k_0$.

$$H_0 : \theta \in \Omega_0 \qquad H_1 : \theta \in \Omega_1 \backslash \Omega_0$$

to test $H_0 : \theta \in \Omega_0$, we use LR statistic

$$G = 2\log\left(\frac{L_1}{L_0}\right)$$

where $L_j$ is the maximum likelihood value over $\Omega_j$.

for large $n$, the $P$-value can be approximately computed, because:

if $\theta \in \Omega_0$, as $n \to \infty$,
the distribution of $G$ converges to $\chi^2_{k_1 - k_0}$

## Normal LR test

$x_1, \dots, x_n$ are form iid $N(\mu, \sigma^2)$ RVs. to test $H_0 : \mu = 0$:

| $\sigma$ | $\Omega_1$ | $k_1$ | $\Omega_0$ | $k_0$ |
|---|---|---|---|---|
| known | $\mathbb{R}$ | 1 | $\{0\}$ | 0 |
| unknown | $\mathbb{R} \times \mathbb{R}_+$ | 2 | $\{0\} \times \mathbb{R}_+$ | 1 |

under $H_0$, for large $n$, approximately $G \sim \chi_1^2$

- **case 1**: $\sigma$ known
  - $\Omega_1 : \log L_1 = -\frac{n\hat{\sigma}^2}{2\sigma^2}$
  - $\Omega_0 : \log L_0 = -\frac{n\hat{\mu}^2}{2\sigma^2}$
  - $G = 2(\log L_1 - \log L_0) = \frac{n\bar{X}^2}{\sigma^2}$
    - if $H_0$ holds ($\mu = 0$), then $\bar{X} \sim N(0, \frac{\sigma^2}{n})$. for any $n$, $G \sim \chi_1^2$ exactly.
- **case 2**: $\sigma$ unknown
  - $\Omega_1 : \log L_1 = -\frac{n}{2}\log\hat{\sigma}^2 - \frac{n}{2}$
  - $\Omega_0 : \log L_0 = -\frac{n}{2}\log\hat{\mu}_2 - \frac{n}{2}$
  - $G = 2(\log L_1 - \log L_0) = n\log(\frac{\hat{\mu}_2}{\hat{\sigma}^2})$
  - if $H_0$ holds ($\mu = 0$), for large $n$, $G \sim \chi_1^2$ approximately

## Summary

- LR test applies when the investigator wants to know the goodness-of-fit of a model relative to a larger model, of dimensions $k_0 < k_1$.
- test statistic, $G = 2\log\left(\frac{L_1}{L_0}\right)$
  - $L_0, L_1$ are the maximum likelihood value under the small and large models
- if $n$ is large, the $P$-value $\Pr(G > g)$ (computed provided $H_0$ is true) can be approximated by a $\chi^2_{k_1 - k_0}$ distribution