# ST2131
AY21/22 SEM 2
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# 01. COMBINATORIAL ANALYSIS

## The Basic Principle of Counting

- **basic principle of counting** $\rightarrow$ Suppose that two experiments are performed. If exp1 can result in any one of $m$ possible outcomes and if, for each outcome of exp1, there are $n$ possible outcomes of exp2, then together there are $mn$ possible outcomes of the two experiments.
- **generalized basic principle of counting** $\rightarrow$ If $r$ experiments are performed such that the first one may result in any of $n_1$ possible outcomes and if for each of these $n_1$ possible outcomes, there are $n_2$ possible outcomes of the 2nd exp, and if ..., then there is a total of $n_1 \cdot n_2 \cdot \dots \cdot n_r$ possible outcomes of $r$ experiments.

## Permutations

**factorials** - $1! = 0! = 1$
**N1** - if we know how to count the number of different ways that an event can occur, we will know the probability of the event.
**N2** - there are $n!$ different arrangements for $n$ objects.
**N3** - there are $\frac{n!}{n_1!\,n_2!\dots n_r!}$ different arrangements of $n$ objects, of which $n_1$ are alike, $n_2$ are alike, ..., $n_r$ are alike.

## Combinations

$$\binom{n}{r} = \frac{n!}{(n-r)!\,r!} = \binom{n-1}{r-1} + \binom{n-1}{r}, \quad 1 \le r \le n$$

**N5** - **The Binomial Theorem**

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

## Multinomial Coefficients

$$\binom{n}{n_1, n_2, \dots, n_r} = \frac{n!}{n_1!\,n_2!\dots n_r!}$$

**N6** - represents the number of possible divisions of $n$ distinct objects into $r$ distinct groups of respective sizes $n_1, n_2, \dots, n_3$, where $n_1 + n_2 + \dots + n_r = n$
**N7** - **The Multinomial Theorem**: $(x_1 + x_2 + \dots + x_r)^n$
$= \sum_{(n_1,\dots,n_r):n_1+n_2+\dots+n_r=n} \frac{n!}{n_1!\,n_2!\dots n_r!} x_1^{n_1} x_2^{n_2} \dots x_r^{n_r}$

## Number of Integer Solutions of Equations

**N8** - there are $\binom{n-1}{r-1}$ distinct *positive* integer-valued vectors $(x_1, x_2, \dots, x_r)$ satisfying $x_1 + x_2 + \dots + x_r = n, \quad x_i > 0, \quad i = 1, 2, \dots, r$
**N9** - there are $\binom{n+r-1}{r-1}$ distinct *non-negative* integer-valued vectors $(x_1, x_2, \dots, x_r)$ satisfying $x_1 + x_2 + \dots + x_r = n$
*Proof.* let $y_k = x_k + 1 \Rightarrow y_1 + y_2 + \dots + y_r = n + r$

# 02. AXIOMS OF PROBABILITY

## Sample Space and Events

- **sample space** $\rightarrow$ The *set* of all outcomes of an experiment
- **event** $\rightarrow$ Any *subset* of the sample space
- **complement** of $E \rightarrow E^c$ is the event that contains all outcomes that are *not* in $E$.
- **subset** $\rightarrow E \subset F$ is all of the outcomes in $E$ that are also in $F$.
  - $E \subset F \wedge F \subset E \Rightarrow E = F$

**DeMorgan's Laws**: $\left(\bigcup_{i=1}^{n} \mathbf{E_i}\right)^{\mathbf{c}} = \bigcap_{i=1}^{n} \mathbf{E_i^c}$ and $\left(\bigcap_{i=1}^{n} \mathbf{E_i}\right)^{\mathbf{c}} = \bigcup_{i=1}^{n} \mathbf{E_i^c}$

## Axioms of Probability

**definition 1: relative frequency**
$P(E) = \lim_{n\to\infty} \frac{n(E)}{n}$. problems: (1) $\frac{n(E)}{n}$ may not converge when $n \to \infty$.
(2) $\frac{n(E)}{n}$ may not converge to the same value if the experiment is repeated.

### Axioms (definition 2)

Consider an experiment with sample space $S$. For each event $E$ of the sample space $S$, we assume that a number $P(E)$ is defined and satisfies the following 3 axioms:
1. $0 \le P(E) \le 1$ 2. $P(S) = 1$
3. For mutually exclusive events, $P(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} P(E_i)$. same for finite case

**mutually exclusive** $\rightarrow$ events for which $E_i E_j = \emptyset$ when $i \ne j$

## Simple Propositions

**N1** - $P(\emptyset) = 0$
**N6** - **probability function** $\Longleftrightarrow$ it satisfies the 3 axioms.
**N8** - if $E \subset F$, then $P(E) \le P(F)$
**N10** - Inclusion-Exclusion identity where $n = 3$
$P(E \cup F \cup G) = P(E) + P(F) + P(G) - P(EF) - P(EG) - P(FG) + P(EFG)$
**N11** - **Inclusion-Exclusion identity** -
$P(E_1 \cup E_2 \cup \dots \cup E_n) = \sum_{i=1}^{n} P(E_i) - \sum_{i_1<i_2} P(E_{i_1} E_{i_2}) + \dots$
$+ (-1)^{r+1} \sum_{i_1<i_2<\dots<i_r} P(E_{i_1} E_{i_2} \dots E_{i_r}) + \dots + (-1)^{n+1} P(E_1 E_2 \dots E_n)$

(i) $P(\bigcup_{i=1}^{n} E_i) \le \sum_{i=1}^{n} P(E_i)$ (based on Inclusion-Exclusion identity)

(ii) $P(\bigcup_{i=1}^{n} E_i) \ge \sum_{i=1}^{n} P(E_i) - \sum_{j<i} P(E_i E_j)$

(iii) $P(\bigcup_{i=1}^{n} E_i) \le \sum_{i=1}^{n} P(E_i) - \sum_{j<i} P(E_i E_j) + \sum_{k<j<i} P(E_i E_j E_k)$

(iv) and so on.

## Sample Space having Equally Likely Outcomes

Consider an experiment with sample space $S = \{e_1, e_2, \dots, e_n\}$. Then $P(\{e_1\}) = P(\{e_2\}) = \dots = P(\{e_n\}) = \frac{1}{n}$ or $P(\{e_i\}) = \frac{1}{n}$.
**N1** - for any event $E$, $P(E) = \frac{\text{\# of outcomes in } E}{\text{\# of outcomes in } S} = \frac{\text{\# of outcomes in } E}{n}$
**increasing sequence** of events $\{E_n, n \ge 1\} \rightarrow E_1 \subset E_2 \subset \dots \subset E_n \subset \dots$
**decreasing sequence** of events $\{E_n, n \ge 1\} \rightarrow E_1 \supset E_2 \supset \dots \supset E_n \supset \dots$

increasing: $\lim_{n\to\infty} E_n = \bigcup_{i=1}^{\infty} E_i$ decreasing: $\lim_{n\to\infty} E_n = \bigcap_{i=1}^{\infty} E_i$
**N2** - for both *increasing* and *decreasing* sequence, $\lim_{n\to\infty} P(E_n) = P(\lim_{n\to\infty} E_n)$

# 03. CONDITIONAL PROBABILITY AND INDEPENDENCE

## Conditional Probability

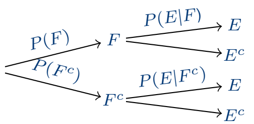
if $P(F) > 0$, then $P(E|F) = \frac{P(E \cap F)}{P(F)}$
**multiplication rule**:
$$P(E_1 \dots E_n) = P(E_1) P(E_2|E_1) P(E_3|E_1E_2) \dots P(E_n|E_1E_2 \dots E_{n-1})$$

**N3** - **axioms of probability** apply to conditional probability
1. $0 \le P(E|F) \le 1$ 2. $P(S|F) = 1$ where $S$ is the sample space
3. If $E_i$ $(i \in \mathbb{Z}_{\ge 1})$ are mutually exclusive, then $P(\bigcup_{1}^{\infty} E_i | F) = \sum_{1}^{\infty} P(E_i|F)$

**N4** - If we define $Q(E) = P(E|F)$, then all previously proven results apply.
- $P(E_1 \cup E_2 | F) = P(E_1|F) + P(E_2|F) - P(E_1E_2|F)$

## Total Probability & Bayes' Theorem

**conditioning formula** - $P(E) = P(E|F)P(F) + P(E|F^c)P(F^c)$




$P(F|E) = \frac{P(EF)}{P(E)} = \frac{P(F) \cdot P(E|F)}{P(E)}$

$P(F^c|E) = \frac{P(EF^c)}{P(E)} = \frac{P(F^c) \cdot P(E|F^c)}{P(E)}$

### Total Probability

**theorem of total probability** - Suppose $F_1, F_2, \dots, F_n$ are mutually exclusive events such that $\bigcup_{i=1}^{n} F_i = S$, then $P(E) = \sum_{i=1}^{n} P(EF_i) = \sum_{i=1}^{n} P(F_i) P(E|F_i)$

### Bayes Theorem

$$P(F_j|E) = \frac{P(EF_j)}{P(E)} = \frac{P(F_j) P(E|F_j)}{\sum_{i=1}^{n} P(F_i) P(E|F_i)}$$

**application of bayes' theorem**

$$P(B_1 \mid A) = \frac{P(A|B_1) \cdot P(B_1)}{P(A|B_1) \cdot P(B_1) + P(A|B_2) \cdot P(B_2)}$$

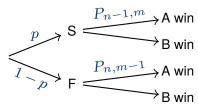
Let $A$ be the event that the person test positive for a disease.
$B_1$: the person has the disease. $B_2$: the person does not have the disease.

| | |
|---|---|
| true positives: $P(B_1 \mid A)$ | false negatives: $P(\bar{A} \mid B_1)$ |
| false positives: $P(A \mid B_2)$ | true negatives: $P(\bar{A} \mid B_2)$ |

## Independent Events

**N1** - $E$ and $F$ are independent $\Longleftrightarrow P(EF) = P(E) \cdot P(F)$
**N2** - $E$ and $F$ are independent $\Longleftrightarrow P(E|F) = P(E)$
**N3** - $E$ and $F$ are independent $\Longleftrightarrow E$ and $F^c$ are independent.
**N4** - if $E, F, G$ are independent, then $E$ will be independent of any event formed from $F$ and $G$. (e.g. $F \cup G$)
**N6** - ($E$ and $F$ are indep) $\wedge$ ($E$ and $G$ are indep) $\not\Rightarrow E$ and $FG$ are independent
**N7** - For independent trials with probability $p$ of success, probability of $m$ successes before $n$ failures, for $m, n \ge 1$,

*method 1*




*method 2*

$$P_{n,m} = \sum_{k=n}^{m+n-1} \binom{m+n-1}{k} p^k (1-p)^{m+n-1-k}$$
$$= P(\ge n \text{ successes in } m+n-1 \text{ trials})$$

# 04. RANDOM VARIABLES

- **random variable** $\rightarrow$ a real-valued function defined on the sample space

## Types of Random Variables

| r.v. | - | $E(X)$ |
|---|---|---|
| binomial | $X$ = # of successes in $n$ trials w/ replacement | $np$ |
| negative binomial | $X$ = # of trials until $k$ successes | $k/p$ |
| geometric | $X$ = # of trials until a success | $1/p$ |
| hypergeometric | $X$ = # of successes in $n$ trials, no replacement | $rn/N$ |

- $X$ is a **Bernoulli r.v.** with parameter $p$ if $\rightarrow$
$$p(x) = \begin{cases} p, & x = 1, \text{ ('success')} \\ 1-p, & x = 0 \quad \text{('failure')} \end{cases}$$
- $Y$ is a **Binomial r.v.** with parameters $n$ and $p \rightarrow \quad Y = X_1 + X_2 + \dots + X_n$ where $X_1, X_2, \dots, X_n$ are independent Bernoulli r.v.'s with parameter $p$.
  - $P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$
  - $P(k$ successes from $n$ independent trials each with probability $p$ of success$)$
  $E(Y) = np, \quad Var(Y) = np(1-p)$
- **Negative Binomial** $\rightarrow X$ = number of trials until $k$ successes are obtained
  - e.g. number of balls drawn (with replacement) until $k$ red balls are obtained
- **Geometric** $\rightarrow X$ = number of trials until a success is obtained
  - $P(X = k) = (1-p)^{k-1} \cdot p$ where $k$ is the number of trials needed
  - e.g. number of balls drawn (with replacement) until 1 red ball is obtained
- **Hypergeometric** $\rightarrow X$ = number of trials until success, *without replacement*
  - $P(X = k) = \frac{\binom{m}{k}\binom{N-m}{n-k}}{\binom{N}{n}}, k = 0, 1, \dots, n$ (for $m$ red balls of $N$ balls)
  - e.g. number of red balls out of $n$ balls drawn without replacement

## Properties

**N1** - if $X \sim$ Binomial$(n,p)$, and $Y \sim$ Binomial$(n-1,p)$, then
$$E(X^k) = np \cdot E[(Y+1)^{k-1}]$$
**N2** - if $X \sim$ Binomial$(n,p)$, then for $k \in \mathbb{Z}^+$,
$$P(X=k) = \frac{(n-k+1)p}{k(1-p)} \cdot P(X=k-1)$$

## Coupon Collector Problem

*Q.* Suppose there are $N$ distinct types of coupons. If $T$ denotes the number of coupons needed to be collected for a complete set, what is $P(T=n)$?

*A.* $P(T>n-1) = P(T \geq n) = P(T=n) + P(T>n)$

$\Rightarrow P(T=n) = P(T>n-1) - P(T>n)$

Let $A_j = \{$no type $j$ coupon is contained among the first $n\}$

$P(T>n) = P(\bigcup_{j=1}^{N} A_j)$

$P(T>n) = \sum_j P(A_j)$ - coupon $j$ is not among the first $n$ collected

$- \sum\sum_{j_1<j_2} P(A_{j_1}A_{j_2})$ - coupon $j_1$ and $j_2$ are not the first $n$

$+ \cdots + (-1)^{N+1} P(A_1 A_2 \cdots A_N)$ by *inclusion-exclusion identity*

$P(A_{j_1}A_{j_2}\cdots A_{j_k}) = (\frac{N-k}{N})^n$

Hence $P(T>n) = \sum_{i=1}^{N-1} \binom{N}{i} (\frac{N-i}{N})^n (-1)^{i+1}$

## Probability Mass Function

**probability mass function**, pmf of $X$ (*discrete*) $\rightarrow$ $p(a) = P(X=a)$

- if $X$ assumes one of the values $x_1, x_2, \ldots$, then $\sum_{i=1}^{\infty} p(x_i) = 1$

## Cumulative Distribution Function

- **cumulative distribution function (cdf)** of a r.v. $X \rightarrow$ the function $F$ defined by
  $F(x) = P(X \leq x), \quad -\infty < x < \infty$
  - $F(x)$ is defined on the entire real line. (aka *distribution function*)

**pmf**,

| $a$ | 1 | 2 | 4 |
|---|---|---|---|
| $p(a)$ | $\frac{1}{2}$ | $\frac{1}{4}$ | $\frac{1}{4}$ |

$F(a) = \sum p(x)$ for all $x \leq a$

**cdf**, $F(a) = \begin{cases} 0, & a<1 \\ 1/2, & 1 \leq a < 2 \\ 3/4, & 2 \leq a < 4 \\ 1, & a \geq 4 \end{cases}$

## Expected Value

- aka population mean/sample mean, $\mu$

*discrete*: $E(X) = \sum_x x \cdot p(x)$

*continuous*: $E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$

$$E[g(x)] = \int_{-\infty}^{\infty} g(x) f(x)\,dx$$

**N1** - if $a$ and $b$ are constants, then $E(aX+b) = aE(X)+b$

**N3** - for a non-negative r.v. $Y$, $E(Y) = \int_0^{\infty} P(Y>y)\,dy$

*Proof.* $\int_0^{\infty} P(Y>y)\,dy = \int_0^{\infty}\int_y^{\infty} f_Y(x)\,dx\,dy$ (because $f(x) = \frac{d}{dx}F(x)$)

$= \int_0^{\infty} x f_Y(x)\,dx \quad = E(Y)$

- $I$ is an indicator variable for event $A$ if $I = \begin{cases} 1, \text{ if } A \text{ occurs} \\ 0, \text{ if } A^c \text{ occurs} \end{cases}$. then $E(I) = P(A)$.

### finding expectation of f(x)

- method 1, using pmf of $Y$: let $Y = f(X)$. Find corresponding $X$ for each $Y$.
- method 2, using pmf of $X$: $E[f(x)] = \sum_x f(x)p(x)$

## Variance

If $X$ is a r.v. with mean $\mu = E[X]$, then the **variance** of $X$ is defined by
$$Var(X) = E[(X-\mu)^2]$$
$$= E(x^2) - [E(x)]^2$$

- $Var(aX+b) = a^2 Var(x)$
- $Var(X) = \sum_{x_i}(x_i-\mu)^2 \cdot p(x_i)$ (deviation · weight)

## Poisson Random Variable

a r.v. $X$ is said to be a **Poisson r.v.** with parameter $\lambda$ if for some $\lambda > 0$,
$$P(X=i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$
$$E(X) = \lambda, \quad Var(X) = \lambda$$

- $\sum_{i=0}^{\infty} P(X=i) = 1$
- **Poisson Approximation of Binomial** - if $X \sim$ Binomial$(n,p)$, where $n$ is large and $p$ is small, then $X \dot\sim$ Poisson$(\lambda)$ where $\lambda = np$.
  - ✓ *weak* dependence is ok
- **2 ways** to look at the Poisson distribution
  1. an approximation to the binomial distribution with large $n$ and small $p$
  2. counting the number of events that occur at *random* at certain points in time

## Poisson distribution as random events

Let $N(t)$ be the number of events that occur in time interval $[0,t]$.

**N1** - If the 3 assumptions are true, then $N(t) \sim$ Poisson$(\lambda t)$.

**N2** - If $\lambda$ is the *rate of occurrences* of events per unit time, then the number of occurrences in an interval of length $t$ has a Poisson distribution with mean $\lambda t$.

$$P(N(t)=k) = \frac{e^{-\lambda t}(\lambda t)^k}{k!}, \text{ for } k \in \mathbb{Z}_{\geq 0}$$

### o(h) notation

$o(h)$ stands for any function $f(h)$ such that $\lim_{h\to 0}\frac{f(h)}{h} = 0$

- $o(h) + o(h) = o(h)$
- $\frac{\lambda t}{n} + o(\frac{t}{n}) \dot= \frac{\lambda t}{n}$ for large $n$

## Expected Value of sum of r.v.

For a r.v. $X$, let $X(s)$ denote the value of $X$ when $s \in \mathcal{S}$

**N1** - $E(x) = \sum_i x_i P(X=x_i) = \sum_{s\in\mathcal{S}} X(s)p(s)$ where $\mathcal{S}_i = \{s : X(s) = x_i\}$

**N2** - $E(\sum_{i-1}^{n}) = \sum_{i=1}^{n} E(X_i)$ for r.v. $X_1, X_2, \ldots, X_n$

### e.g. distribution of time to next event

Q. suppose an accident happens at a rate of 5 per day. Find the distribution of time, starting from now, until the next accident.

*A.* Let $X =$ time (in days) until the next accident.
Let $V =$ be the number of accidents during time period $[0,t]$.

$V \sim$ Poisson$(5t)$ $\quad \Rightarrow P(V=k) = \frac{e^{-5t}\cdot(5t)^k}{k!}$

$P(X>t) = P($no accidents happen during $[0,t]) = P(V=0) = e^{-5t}$

$P(X \leq t) - 1 - e^{-5t}$

### example: finding pdf

*Q* - Find the pdf of $(b-a)X+a$ where $a, b$ are constants, $b>a$. The pdf of $X$ is given by $f(x) = \begin{cases} 1, & 0 \leq X \leq 1 \\ 0, & \text{otherwise} \end{cases}$.

*A.* Let $Y = (b-a)X + a$.

cdf, $F_Y(y) = P(Y \leq y) = P((b-a)X + a \leq y) = P(X \leq \frac{y-a}{b-a})$

$F_Y(y) = \int_0^{\frac{y-a}{b-a}} 1\,dx = \frac{y-a}{b-a}, \quad a<y<b$

$f_Y(y) = \frac{d}{dy}F_Y(y) = \begin{cases} \frac{1}{b-a}, & a<y<b \\ 0, & \text{otherwise} \end{cases}$

# 05. CONTINUOUS RANDOM VARIABLES

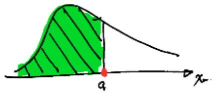
$X$ is a **continuous r.v.** $\rightarrow$ if there exists a nonnegative function $f$ defined for all real $x \in (-\infty, \infty)$, such that $P(X \in B) = \int_B f(x)\,dx$

**N1** - $P(X \in (-\infty,\infty)) = \int_{-\infty}^{\infty} f(x)\,dx = 1$

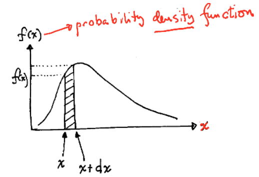
**N2** - $P(a \leq X \leq b) = \int_a^b f(x)\,dx$

**N3** - $P(X=a) = \int_a^a f(x)\,dx = 0$

**N4** - $P(X<a) = P(X \leq a) = \int_{-\infty}^{a} f(x)\,dx$




**N5** - interpretation of **probability density function**




$$P(x < X < x+dx) = \int_x^{x+dx} f(y)\,dy$$
$$\approx f(x)\cdot dx$$

pdf at $x$, $f(x) \approx \frac{P(x<X<x+dx)}{dx}$

**N6** - if $X$ is a continuous r.v. with pdf $f(x)$ and cdf $F(x)$, then $f(x) = \frac{d}{dx}F(x)$. (*Fundamental Theorem of Calculus*)

**N7** - median of $X$, $x$ occurs where $F(x) = \frac{1}{2}$

## Generating a Uniform r.v.

if $X$ is a continuous r.v. with cdf $F(x)$, then

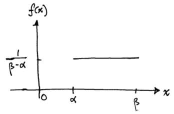
- **N8** - $F(X) = U \sim uniform(0,1)$.

  *Proof.* let $Y = F(X)$. then cdf of $Y$, $F_Y(y)$ =
  $P(Y \leq y) = P(F(X) \leq y) = P(X \leq F^{-1}(y)) = F(F^{-1}(y)) = y$.

- **N9** - $X = F^{-1}(U) \sim$ cdf $F(x)$.
  - generating a r.v. from a uniform(0, 1) r.v. and a r.v. with cdf $F(x)$.

## Uniform Random Variable

$X$ is a **uniform r.v.** on the interval $(\alpha,\beta)$, $X \sim Uniform(\alpha,\beta)$ if its pdf is given by

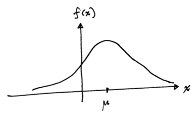



$$f(x) = \begin{cases} \frac{1}{\beta-\alpha}, & \alpha < x < \beta \\ 0, & \text{otherwise} \end{cases}$$
$$E(X) = \frac{\alpha+\beta}{2}, \quad Var(X) = \frac{(\beta-\alpha)^2}{12}$$

if $X \sim Uniform(\alpha,\beta)$, then $\frac{x-\alpha}{\beta-\alpha} \sim Uniform(0,1)$

## Normal Random Variable

$X$ is a **normal r.v.** with parameters $\mu$ and $\sigma^2$, $X \sim N(\mu,\sigma^2)$ if the pdf of $X$ is given by
$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(\frac{x}{\mu}\sigma)^2}, \quad -\infty < x < \infty$$
$$E(x) = \mu, \quad Var(X) = \sigma^2$$




if $X \sim N(\mu,\sigma^2)$, then $\frac{X-\mu}{\sigma} \sim N(0,1)$

if $Y \sim N(\mu,\sigma^2)$ and $a$ is a constant, $F_y(a) = \Phi(\frac{a-\mu}{\sigma})$

**standard normal distribution** $\rightarrow X \sim N(0,1)$

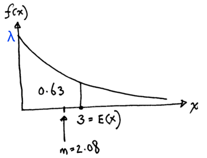
- $F(x) = P(X \leq x) = \frac{1}{\sqrt{r\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}y^2}\,dy = \Phi(x)$

## Normal Approximation to the Binomial Distribution

if $S_n \sim Binomial(n,p)$, then $\frac{S_n - np}{\sqrt{np(1-p)}} \sim N(0,1)$ for large $n$.
$$\mu = np, \quad \sigma^2 = np(1-p)$$

## Exponential Random Variable

a *continuous* r.v. $X$ is a **exponential r.v.**, $X \sim Exponential(\lambda)$ or $Exp(\lambda)$
if for some $\lambda > 0$, its pdf is given by




$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$E(X) = \frac{1}{\lambda}, \quad Var(X) = \frac{1}{\lambda^2}$$

$$P(X < a) = \int_0^a \lambda e^{-\lambda x}\, dx$$

- an exponential r.v. is *memoryless*.
  - a non-negative r.v. is **memoryless** $\rightarrow$ if
    $P(X > s + t \mid X > t) = P(X > s)$ for all $s, t > 0$.

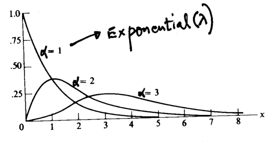
## Gamma Distribution

a r.v. $X$ has a **gamma distribution**, $X \sim Gamma(\alpha, \lambda)$
with parameters $(\alpha, \lambda)$, $\lambda > 0$ and $\alpha > 0$ if its pdf is given by

$$f(x) \begin{cases} \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$$E(X) = \frac{\alpha}{\lambda} \quad Var(X) = \frac{\alpha}{\lambda^2}$$

where the gamma function $\Gamma(\alpha)$ is defined as $\Gamma(\alpha) = \int_0^\infty e^{-y} y^{\alpha-1}\, dy$.




Gamma densities (λ = 1).

**N1** - $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$

*Proof.* using integration by parts of LHS to RHS

**N2** - if $\alpha$ is an integer $n$, then $\Gamma(n) = (n-1)!$

**N3** - if $X \sim Gamma(\alpha, \lambda)$ and $\alpha = 1$, then $X \sim Exp(\lambda)$.

**N4** - for events occurring randomly in time following the 3 assumptions of poisson distribution, the **amount of time elapsed** until a total of $n$ events has occurred is a gamma r.v. with parameters $(n, \lambda)$.

- time at which the $n$-th event occurs, $T_n \sim Gamma(n, \lambda)$
- number of events in time period $[0, t]$, $N(t) \sim Poisson(\lambda t)$

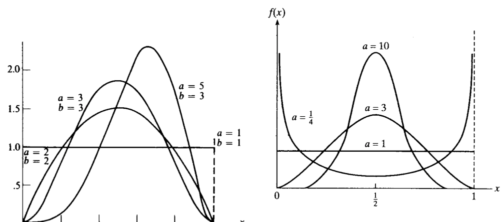
**N5** - $Gamma(\alpha = \frac{n}{2}, \lambda = \frac{1}{2}) = \chi_n^2$ (chi-square distribution to $n$ degrees of freedom)

## Beta Distribution

a r.v. $X$ is said to have a **beta distribution**, $X \sim Beta(a, b)$

$$f(x) = \begin{cases} \frac{1}{\beta(a,b)} x^{a-1}(1-x)^{b-1}, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

$$E(X) = \frac{a}{a+b} \quad Var(X) = \frac{ab}{(a+b)^2(a+b+1)}$$




Beta densities with parameters $(a, b)$ when $a = b$.

**N1** - $\beta(a, b) = \int_0^1 x^{a-1}(1-x)^{b-1}\, dx$
**N2** - $\beta(a = 1, b = 1) = Uniform(0, 1)$
**N3** - $\beta(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$

## Cauchy Distribution

a r.v. $X$ has a cauchy distribution, $X \sim Cauchy(\theta)$
with parameter $\theta$, $\infty < \theta < \infty$ if its density is given by

$$f(x) = \frac{1}{\pi} \cdot \frac{1}{1+(x-\theta)^2}, -\infty < x < \infty$$

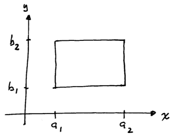
# 06. JOINTLY DISTRIBUTED RANDOM VARIABLES

## Joint Distribution Function

the **joint cumulative distribution function** of the pair of r.v. $X$ and $Y$ is $\rightarrow$
$F(x, y) = P(X \leq x, Y \leq y), -\infty < x < \infty, \ -\infty < y < \infty$

**N1** - **marginal cdf of $X$**, $F_X(x) = \lim_{y \to \infty} F(x, y)$.

**N2** - **marginal cdf of $Y$**, $F_Y(y) = \lim_{x \to \infty} F(x, y)$.




**N3** - $P(X > a, Y > b) = 1 - F_X(a) - F_Y(b) + F(a, b)$
**N4** - $P(a_1 < X \leq a_2, b_1 < Y \leq b_2)$
$= F(a_2, b_2) + F(a_1, b_1) - F(a_1, b_2) - F(a_2, b_1)$

## Joint Probability Mass Function

if $X$ and $Y$ are both discrete r.v., then their **joint pmf** is defined by
$p(i, j) = P(X = i, Y = j)$

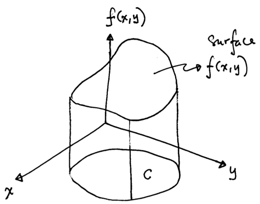
**N1** - **marginal pmf of $X$**, $P(X = i) = \sum_j P(X = i, Y = j)$
**N2** - **marginal pmf of $Y$**, $P(Y = i) = \sum_i P(X = i, Y = j)$

## Joint Probability Density Function

the r.v. $X$ and $Y$ are said to be *jointly continuous* if there is a function $f(x, y)$ called the **joint pdf**, such that for any two-dimensional set $C$,

$$P[(X, Y) \in C] = \iint_C f(x, y)\, dx\, dy$$

$=$ volume under the surface over the region $C$.

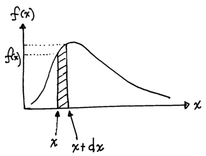



**N1** - if $C = \{(x, y) : x \in A, y \in B\}$, then
$P(X \in A, Y \in B) = \int_B \int_A f(x, y)\, dx\, dy$

**N2** - $F(a, b)$
$= P\Big(X \in (-\infty, a], Y \in (-\infty, b]\Big)$
$= \int_{-\infty}^b \int_{-\infty}^a f(x, y)\, dx\, dy$

**N3** - $f(a, b) = \frac{\delta^2}{\delta a \delta b} F(a, b)$

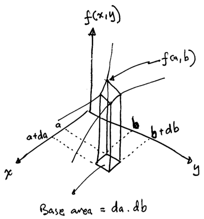
### interpretation of pdf




$$P(x < X < x + dx) = \int_x^{x+dx} f(y)\, dy$$
$$\approx f(x)\, dx$$

pdf at $x$, $f(x) \approx \frac{P(x < X < x + dx)}{dx}$

### interpretation of joint pdf



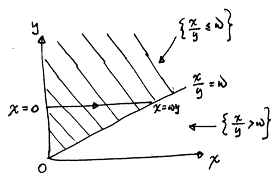

$P(a < X < a + da, b < Y < b + db)$
$= \int_b^{b+db} \int_a^{a+da} f(x, y)\, dx\, dy$
$\approx f(a, b)\, da\, db$ (density of probability)

**marginal pdf of $X$**, $f_X(x) = \int_{-\infty}^\infty f(x, y)\, dy$
**marginal pdf of $Y$**, $f_Y(x) = \int_{-\infty}^\infty f(x, y)\, dx$

### how to do a double integral

**example** - given the joint pdf of $X$ and $Y$, find the pdf of r.v. $X/Y$.

*ans.* set dummy variable $W = X/Y$, then
$F_W(w) = P(W \leq w) = P(\frac{X}{Y} \leq w)$
$P(\frac{X}{Y} \leq w) = \int_0^\infty \int_0^{wy} e^{-x-y}\, dx\, dy$




## Independent Random Variables

**N1** - $X, Y$ are **independent** $\rightarrow P(X \in A, Y \in B) = P(X \in A) \cdot P(Y \in B)$

**N2** - $X$ and $Y$ are **independent** $\rightarrow P(X \leq a, Y \leq b) = P(X \leq a) \cdot P(Y \leq b)$
or $F(a, b) = F_X(a) \cdot F_Y(b)$ $\Rightarrow$ joint cdf is the product of the marginal cdfs

**N3** - *discrete case*: discrete r.v. $X$ and $Y$ are **independent** $\Longleftrightarrow$
$P(X = x, Y = y) = P(X = x) \cdot P(Y = y)$ for all $x, y$.

**N4** - *continuous case*: jointly continuous r.v. $X$ and $Y$ are **independent** $\Longleftrightarrow$
$f(x, y) = f_X(x) \cdot f_Y(y)$ for all $x, y$.
$-\infty < x < \infty, \quad -\infty < y < \infty$

**N5** - independence is a **symmetric** relation $\rightarrow X$ indep of $Y \Longleftrightarrow Y$ indep of $X$

if $X$ and $Y$ are independent, then for any functions $h$ and $g$,
$E[g(X)h(Y)] = E[g(X)] \cdot E[h(Y)]$

## Sum of Independent Random Variables

**N1** - for independent, continuous r.v. $X$ and $Y$ having pdf $f_X$ and $f_Y$,
$F_{X+Y}(a) = \int_{-\infty}^\infty F_X(a - y) f_Y(y)\, dy$
$f_{X+Y}(a) = \int_{-\infty}^\infty f_X(a - y) f_Y(y)\, dy$

### Distribution of Sums of Independent r.v.

for $i = 1, 2, \ldots, n$,

1. $X_i \sim Gamma(t_i, \lambda) \quad \Rightarrow \quad \sum_{i=1}^n X_i \sim Gamma(\sum_{i=1}^n t_i, \lambda)$
2. $X_i \sim Exp(\lambda) \quad \Rightarrow \quad \sum_{i=1}^n X_i \sim Gamma(n, \lambda)$
3. $Z_i \sim N(0, 1) \quad \Rightarrow \quad \sum_{i=1}^n z_i^2 \sim \chi_n^2 = Gamma(\frac{n}{2}, \frac{1}{2})$
4. $X_i \sim N(\mu_i, \sigma_i^2) \quad \Rightarrow \quad \sum_{i=1}^n X_i \sim N(\sum_{i=1}^n \mu_i, \sum_{i=1}^n \sigma_i^2)$
5. $X \sim Poisson(\lambda_1), Y \sim Poisson(\lambda_2) \Rightarrow X + Y \sim Poisson(\lambda_1 + \lambda_2)$
6. $X \sim Binom(n, p), Y \sim Binom(m, p) \Rightarrow X + Y \sim Binom(n + m, p)$

### Conditional Distribution (discrete)

for discrete r.v. $X$ and $Y$, the **conditional pmf** of $X$ given that $Y = y$ is
$$P_{X|Y}(x|y) = P(X = x|Y = y) = \frac{P(X=x, Y=y)}{P(Y=y)} = \frac{p(x,y)}{p_Y(y)}$$

for discrete r.v. $X$ and $Y$, the **conditional pdf** of $X$ given that $Y = y$ is
$$F_{X|Y}(x|y) = P(X \leq x|Y = y) = \sum_{a \leq x} P(X = a, Y = y) = \sum_{a \leq x} P_{X|Y}(a|y)$$

**N0** - equivalent notation:

- $P_{X|Y}(x|y) = P(X = x|Y = y)$
- $P_X(x) = P(X = x)$

**N1** - if $X$ is independent of $Y$, then $P_{X|Y}(x|y) = P_X(x)$

### Conditional Distribution (continuous)

for $X$ and $Y$ with joint pdf $f(x, y)$, the **conditional pdf** of $X$ given that $Y = y$ is
$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} \quad \text{for all } y \text{ s.t. } f_Y(y) > 0$$
$$F_{X|Y}(a|y) = P(X \leq a|Y = y) = \int_{-\infty}^a f_{X|Y}(x|y)\, dx$$

**N1** - for any set $A$, $P(X \in A|Y = y) = \int_A f_{X|Y}(x|y)\, dy$

**N2** - if $X$ is independent of $Y$, then $f_{X|Y}(x|y) = f_X(x)$.

**!** "find the marginal/conditional pdf of $Y$" $\Rightarrow$ must include the **range** too!!

### Joint Probability Distribution of Functions of r.v.

Let $X_1$ and $X_2$ be jointly continuous r.v. with joint pdf $f_{x_1,x_2}(x_1,x_2)$. Suppose $Y_1 = g_1(X_1, X_2)$ and $Y_2 = g_2(X_1, X_2)$ satisfy

1. the equations $y_1 = g_1(X_1, X_2)$ and $y_2 = g_2(X_1, X_2)$ can be *uniquely* solved for $x_1, x_2$ in terms of $y_1$ and $y_2$
2. $g_1(x_1, x_2)$ and $g_2(x_1, x_2)$ have continuous partial derivatives at all points $(x_1, x_2)$ such that $J(x_1, x_2) = \begin{vmatrix} \frac{\delta g_1}{\delta x_1} & \frac{\delta g_1}{\delta x_2} \\ \frac{\delta g_2}{\delta x_1} & \frac{\delta g_2}{\delta x_2} \end{vmatrix} = \frac{\delta g_1}{\delta x_1} \cdot \frac{\delta g_2}{\delta x_2} - \frac{\delta g_2}{\delta x_1} \cdot \frac{\delta g_1}{\delta x_2} \neq 0$

then

$$f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}(x_1, x_2) \frac{1}{|J(x_1,x_2)|}$$

where $x_1 = h_1(y_1, y_2), x_2 = h_2(y_1, y_2)$

## 07. PROPERTIES OF EXPECTATION

- for a **discrete** r.v. $X$, $E(X) = \sum_x x \cdot p(x) = \sum_x \cdot P(X = x)$
- for a **continuous** r.v. $X$, $E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$
- for a **non-negative integer-valued** r.v. $Y$, $E(Y) = \sum_{i=1}^{\infty} P(Y \geq i)$
- for a **non-negative** r.v. $Y$, $E(Y) = \int_{-\infty}^{\infty} P(Y > y)\, dy$

### Expectations of Sums of Random Variables

for $X$ and $Y$ with joint pmf $p(x, y)$ and joint pdf $f(x, y)$,

$$E[g(x, y)] = \sum_y \sum_x g(x, y) p(x, y)$$

$$E[g(x, y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f(x, y)\, dx\, dy$$

**N2** - if $P(a \leq X \leq b) = 1$, then $a \leq E(X) \leq b$

**N4** - for r.v.s $X$ and $Y$, if $X \geq Y$, then $E(X) \geq E(Y)$

**N5** - let $X_1, \ldots, X_n$ be independent and identically distributed r.v.s having distribution $P(X_i \leq x) = F(x)$ and expected value $E(X_i) = \mu$.

if $\bar{X} = \sum_{i=1}^{n} \frac{X_i}{n}$, then $E(\bar{X}) = \mu$

**N6** - $\bar{X}$ is the **sample mean**. $\Rightarrow$ sample mean = population mean

! trick: express a r.v. as a sum of r.v. with easier to find expectation

### examples

- hypergeometric with $r$ red balls out of $N$ balls with $n$ trials
  - indicator r.v. $= 1$ if the $i$th ball selected is red
  - $P(Y_i = 1) = \frac{r}{N} \Rightarrow E(Y_i) = \frac{r}{N} \Rightarrow E(X) = \sum_{i=1}^{n} Y_i = n\frac{r}{N}$
- coupon collector problem:
  - let $X$ = number of coupons collected for a complete set
  - let $X_i$ = additional number to be collected to obtain distinct type after $i$ distinct types have been collected. $X_i \sim Geometric(p = \frac{N-i}{N})$
  - $E(X) = \sum_{i=1}^{N-1} E(X_i) = 1 + \frac{1}{\frac{N-1}{N}} + \frac{1}{\frac{N-2}{N}} + \cdots + \frac{1}{\frac{1}{N}}$ $= N(\frac{1}{N} + \frac{1}{N-1} + \cdots + 1)$

### Covariance, Variance of Sums and Correlations

**covariance** $\rightarrow$ measure of *linear relationship*

$$Cov(X, Y) = E[(X - E[X])(Y - E[Y])]$$
$$Cov(X, Y) = E(XY) - E(X)E(Y)$$

**N1** - $X$ and $Y$ are independent $\Rightarrow Cov(X, Y) = 0$

**N2** - $Cov(X, Y) = 0 \not\Rightarrow X$ and $Y$ are independent. *Proof.* let $E(X) = 0$, $E(XY) = 0 \Rightarrow Cov(X, Y) = 0$, but not independent e.g. non-linear relationship

### Covariance properties

1. $Cov(X, Y) = Cov(Y, X)$
2. $Cov(X, X) = Var(X)$
3. $Cov(aX, Y) = aCov(X, Y)$
4. $Cov(\sum_{i=1}^{n} X_i, \sum_{j=1}^{m} Y_j) = \sum_{i=1}^{n} \sum_{j=1}^{m} Cov(X_i, Y_j)$
5. $Cov(I_A, I_B) = P(B)[P(A|B) - P(A)]$

for variance:

**N1** - $Var(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} Var(X_i) + 2\sum\sum_{i<j} Cov(X_i, X_j)$

**N2** - if $X_1, \ldots, X_n$ are *pairwise independent*, $Var(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} Var(X_i)$

**N3** - for $n$ independent and identically distributed r.v. with variance $\sigma^2$,

$$\bar{X} = \frac{1}{n} \sum_{i=1}^{n} x_i \qquad S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$$

$$Var(\bar{X}) = \frac{\sigma^2}{n} \qquad E(S^2) = \sigma^2$$

$\Rightarrow S^2$ is an *unbiased estimator* for $\sigma^2$.

### Correlation

**correlation** of two r.v. $X$ and $Y$, $\rho(X, Y) = \frac{Cov(X,Y)}{\sqrt{Var(X) \cdot Var(Y)}}$

**N1** - $-1 \leq \rho(X, Y) \leq 1$ where $-1$ and $1$ denote a perfect negative and positive linear relationship respectively.

**N2** - $\rho(X, Y) = 0 \Rightarrow$ no *linear* relationship - uncorrelated

**N3** - $\rho(X, Y) = 1 \Rightarrow Y = aX + b, a = \frac{\delta y}{\delta x} > 0$

**N4** - for independent events $A, B$ with indicator r.v. $I_A, I_B$: $Cov(I_A, I_B) = 0$.

**N5** - deviation is not correlated with the sample mean. For independent & identically distributed r.v. $X_1, X_2, \ldots, X_n$ with variance $\sigma^2$, then $Cov(X_i - \bar{X}, \bar{X}) = 0$.

### Conditional Expectation

the **conditional expectation** of $X$ given that $Y = y$, $\forall y$ s.t. $P_Y(y) > 0$, is:

$$E[X|Y = y] = \sum_x x \cdot P(X = x|Y = y) = \sum_x x \cdot p_{X|Y}(x|y)$$

$$E(X|Y = y) = \int_{-\infty}^{\infty} x \cdot f_{X|Y}(x|y)\, dx = \int_{-\infty}^{\infty} x \cdot \frac{f(x,y)}{f_Y(y)}\, dx$$

! note the range for $f_{X|Y}(x|y)$

**N1** - If $X, Y \sim Geometric(p)$, then $P(X = i|X + Y = n) = \frac{1}{n-1}$, a uniform distribution.

**N2** - $E(X|X + Y = n) = \sum_{i=1}^{n-1} i \cdot P(X = i|X + Y = n) = \frac{n}{2}$

discrete case: $E[g(x)|Y = y] = \sum_x g(x) P_{X|Y}(x|y)$

continuous case: $E[g(x)|Y = y] = \int_{-\infty}^{\infty} g(x) f_{X|Y}(x|y)\, dx$

then $E(X) = E_{\text{w.r.t. } y}(E_{\text{w.r.t. } X|Y=y}(X|Y))$

### Deriving Expectation

$E(X) = E_Y(E_X(X|Y))$

discrete case: $E(X) = \sum_y E(X|Y = y)P(Y = y)$

continuous case: $E(X) = \int_{-\infty}^{\infty} E(X|Y = y) f_Y(y)\, dy$

**N3** - 3 methods for finding $E(X)$ given $f(x, y)$

1. using $E(g(x, y)) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f(x, y)\, dx\, dy$ $\Rightarrow$ let $g(x, y) = x$
2. using $E(X) = \int_{-\infty}^{\infty} x f_X(x)\, dx$
3. using $E(X) = \int_{-\infty}^{\infty} E(X|Y = y) f_Y(y)\, dy$

**N4** - $E(\sum_{i=1}^{N} X_i) = E_N(E(\sum_{i=1}^{N} X_i|N)) = \sum_{n=0}^{\infty} E(\sum_{i=1}^{N} X_i|N = n) \cdot P(N = n)$

### Computing Probabilities by Conditioning

*discrete*: $P(E) = \sum_y P(E|Y = y)P(Y = y)$

*continuous*: $P(E) = \int_{-\infty}^{\infty} P(E|Y = y) f_Y(y)\, dy$

*Proof.* $X$ is an indicator r.v.; $E(X|Y = y) = P(X = 1|Y = y) = P(E|Y = y)$

**N5** - $P(X < Y) = \int P(X < Y|Y = y) \cdot f_Y(y)$

### Conditional Variance

$$Var(X|Y) = E[(X - E(X|Y))^2 \mid Y]$$
$$= E(X^2|Y) - [E(X|Y)]^2$$

**N6** - $Var(X) = E[Var(X|Y)] + Var[E(X|Y)]$

**N7** - $E(f(Y)) = E(f(Y)|Y = t) = E(f(t)|Y = t)$ $= E(f(t))$ if $N(t)$ and $Y$ are independent

### Moment Generating Functions

**moment generating function** $M(t)$ of the r.v. $X \rightarrow$

$M(t) = E(e^{tX})$ for all real values of $t$

- if $X$ is *discrete* with pmf $p(x)$, $M(t) = \sum_x e^{tx} \cdot p(x)$
- if $X$ is *continuous* with pdf $f(x)$, $M(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$

$M(t)$ is called the **mgf** because *all moments of* $X$ can be obtained by successively differentiating $M(t)$ and then evaluating the result at $t = 0$.

- the $n^{th}$ moment of of $X$ is given as $E(X^n) = \sum_x x^n \cdot p(x)$
  - $M'(0) = E(X)$, $M''(0) = E(X^2)$, $M^n(0) = E(X^n)$, $n \geq 1$
  - $M'(t) = E(X^n e^{tX})$, $n \geq 1$

if $X$ and $Y$ are independent and have mgf's $M_X(t)$ and $M_Y(t)$ respectively,

**N10** - the mgf of $X + Y$ is $M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$

**N11** - if $M_X(t)$ exists and is finite in some region about $t = 0$, then the distribution of $X$ is **uniquely** determined. $M_X(t) = M_Y(t) \iff X = Y$

joint mgf: $E[e^{tX+sY}] = \int\int e^{tx+sy} f(x, y)\, dy\, dx$

### Common mgf's

- $X \sim Normal(0, 1)$, $M(t) = e^{e^2/2}$
- $X \sim Binomial(n, p)$, $M(t) = (pe^t + (1 - p))^n$
- $X \sim Poisson(\lambda)$, $M(t) = \exp[\lambda(e^t - 1)]$
- $X \sim Exp(\lambda)$, $M(t) = \frac{\lambda}{\lambda - t}$

## 08. LIMIT THEOREMS

**Markov's Inequality** $\rightarrow$ if $X$ is a non-negative r.v., $\forall a > 0$, $P(X \geq a) \leq \frac{E(x)}{a}$.

**Chebyshev's inequality** $\rightarrow$ if $X$ is an r.v. with finite mean $\mu$ and variance $\sigma^2$, then for any value of $k > 0$, $P(|X - \mu| \geq k) \leq \frac{\sigma^2}{k^2}$.

**N1** - if $Var(X) = 0$, then $P(X = E[X]) = 1$

**weak law of large numbers** $\rightarrow$ let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed r.v.s, each with finite mean $E[X_i] = \mu$. Then, for any $\epsilon > 0$, $P\{|\frac{X_1 + \cdots + X_n}{n} - \mu| \geq \epsilon\} \rightarrow 0$ as $n \rightarrow \infty$

**central limit theorem** $\rightarrow$ let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed r.v.s each having mean $\mu$ and variance $\sigma^2$. Then the distribution of $\frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}}$ tends to the standard normal as $n \rightarrow \infty$.

- aka: $\frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \rightarrow z \sim N(0, 1)$
- for $-\infty < a < \infty$, as $n \rightarrow \infty$,
  $P(\frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} \leq a) \rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-x^2/2}\, dx = F(a)$ - cdf of $N(0, 1)$

**N2** - Let $Z_1, Z_2, \ldots$ be a sequence of r.v.s with distribution functions $F_{Z_n}$ and moment generating functions $M_{Z_n}, n \geq 1$. Let $Z$ be a r.v. with distribution function $F_Z$ and mgf $M_Z$.

If $M_{Z_n}(t) \rightarrow M_Z(t)$ for all $t$, then $F_{Z_n}(t) \rightarrow F_Z(t)$ for all $t$ at which $F_Z(t)$ is continuous.

**strong law of large numbers** $\rightarrow$ let $X_1, X_2, \ldots$ be a sequence of independent and identically distribution r.v.s, each having finite mean $\mu = E[X_i]$. Then, with probability $1$, $\frac{X_1 + \cdots + X_n}{n} \rightarrow \mu$ as $n \rightarrow \infty$

approximations - $\lim_{n\to\infty} (1 - \frac{\lambda}{n})^n = e^{-\lambda}$