Self Assignment: Sequences and Series, Recursion, and Number Bases

2024

Introduction

This document will help cover some needed fundamentals of sequences, series, recursion, arithmetic and geometric sequences, and number bases. The topics will be explained with step-by-step examples and detailed calculations, which are essential for building mathematical reasoning and computational skills. This document was made for CM1015 at UOL

1 Sequences and Series

A *sequence* is an ordered list of numbers, while a *series* is the sum of the terms in a sequence.

1.1 Arithmetic Sequences

An arithmetic sequence has a constant difference between consecutive terms, known as the *common difference* (d). Each term in an arithmetic sequence is given by:

$$a_n = a_1 + (n-1) \cdot d$$

where:

- a_n is the n-th term,
- a_1 is the first term,
- d is the common difference.

Example: Find the 5th term in the sequence where $a_1 = 3$ and $d = 4$.
Solution:

$$a_5 = a_1 + (5-1) \cdot d$$

$$= 3 + 4 \cdot 4 = 3 + 16 = 19$$

1.2 Sum of an Arithmetic Sequence

The sum of the first n terms (S_n) of an arithmetic sequence is:

$$S_n = \frac{n}{2}(2a_1 + (n-1)d)$$

Example: Find the sum of the first 10 terms of the sequence where $a_1 = 3$ and $d = 4$.

Solution:

$$S_{10} = \frac{10}{2}(2 \cdot 3 + (10-1) \cdot 4)$$
$$= 5(6 + 36) = 5 \cdot 42 = 210$$

1.3 Geometric Sequences

A geometric sequence has a constant ratio between consecutive terms, called the *common ratio* (r). Each term is given by:

$$a_n = a_1 \cdot r^{n-1}$$

Example: Find the 4th term in a sequence where $a_1 = 2$ and $r = 3$.

Solution:

$$a_4 = 2 \cdot 3^{4-1} = 2 \cdot 3^3 = 2 \cdot 27 = 54$$

1.4 Sum of a Geometric Sequence

The sum of the first n terms of a geometric sequence is:

$$S_n = a_1 \frac{1 - r^n}{1 - r} \quad if r \neq 1$$

Example: Find the sum of the first 5 terms of a geometric sequence with $a_1 = 2$ and $r = 3$.

Solution:

$$S_5 = 2 \cdot \frac{1 - 3^5}{1 - 3} = 2 \cdot \frac{1 - 243}{-2} = 2 \cdot \frac{-242}{-2} = 2 \cdot 121 = 242$$

2 Recursion

Recursion defines each term in a sequence based on previous terms. The Fibonacci sequence is a well-known example, where each term is the sum of the two preceding terms:

$$F_n = F_{n-1} + F_{n-2}$$

with initial conditions $F_0 = 0$ and $F_1 = 1$.

Example: Calculate F_5 in the Fibonacci sequence.

Solution:

$$F_2 = F_1 + F_0 = 1 + 0 = 1, F_3 = F_2 + F_1 = 1 + 1 = 2, F_4 = F_3 + F_2 = 2 + 1 = 3, F_5 = F_4 + F_3 = 3 + 2 = 5.$$

3 Number Bases

Numbers are often represented in various bases. We cover binary (base-2) and hexadecimal (base-16) systems.

3.1 Binary Numbers

Binary (base-2) numbers use only 0 and 1. Each position represents a power of 2.

3.1.1 Conversion to Decimal

To convert binary to decimal, sum each bit times its positional power of 2.

$$(1011)_2 = 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 = 8 + 0 + 2 + 1 = 11$$

Example: Convert $(1101)_2$ to decimal.
Solution:

$$(1101)_2 = 1 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 = 8 + 4 + 0 + 1 = 13$$

3.1.2 Binary Addition

Binary addition follows these rules: $0 + 0 = 0$, $1 + 0 = 1$, $1 + 1 = 10$ (carry 1).
Example: Add $(101)_2$ and $(110)_2$.
Solution:

$$\begin{array}{r} 1 \\ 101 \\ +\ 110 \\ \hline 1011 \end{array}$$

3.2 Hexadecimal Numbers

Hexadecimal (base-16) numbers use digits 0-9 and letters A-F, where $A = 10$, $B = 11$, up to $F = 15$.

3.2.1 Conversion to Decimal

To convert hexadecimal to decimal, multiply each digit by its positional power of 16.

$$(1A)_{16} = 1 \cdot 16^1 + 10 \cdot 16^0 = 16 + 10 = 26$$

Example: Convert $(2F)_{16}$ to decimal.
Solution:

$$(2F)_{16} = 2 \cdot 16^1 + 15 \cdot 16^0 = 32 + 15 = 47$$

3.3 Hexadecimal to Binary Conversion

Each hexadecimal digit converts to a 4-bit binary sequence.

Example: Convert $(1A)_{16}$ to binary.

Solution: 1 in hex is $(0001)_2$, and A (10) is $(1010)_2$, so:

$$(1A)_{16} = (0001\ 1010)_2$$