Self Assignment: Basic Probability Concepts II

2024

Continued Introduction

This document will continue what was explained in Basic Probability Concepts I document. The document content will cover topics of Probability distributions and expected Value.

1 Probability Distributions

A **probability distribution** describes how the values of a random variable are distributed or spread out. It provides a way to calculate the probability associated with different possible outcomes of a random variable. There are two primary types of probability distributions: one for **discrete random variables** and another for **continuous random variables**.

1.1 Discrete Probability Distributions

For discrete random variables, the probability distribution is represented by a **probability mass function** (PMF). The PMF gives the probability that a discrete random variable takes on a specific value.

1.1.1 Binomial Distribution

The **binomial distribution** is a discrete probability distribution that describes the number of successes in a fixed number of independent Bernoulli trials, where each trial has the same probability of success.

Let X be a binomial random variable that counts the number of successes in n independent trials, where each trial has a probability of success p. The PMF of the binomial distribution is given by the formula:

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k} \quad \text{for } k = 0, 1, 2, \ldots, n$$

where $\binom{n}{k}$ is the binomial coefficient:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Here:

- n is the number of trials,
- k is the number of successes,
- p is the probability of success in each trial, and
- $1 - p$ is the probability of failure.

Example: Tossing a Coin

Consider tossing a fair coin three times. Let X be the random variable representing the number of heads obtained. The probability of getting heads in each trial is $p = \frac{1}{2}$. The binomial distribution can be used to compute the probabilities of obtaining 0, 1, 2, or 3 heads.

The probability mass function (PMF) for X is given by:

$$P(X = k) = \binom{3}{k} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{3-k}$$

which results in:

$$P(X = 0) = \frac{1}{8}, \quad P(X = 1) = \frac{3}{8}, \quad P(X = 2) = \frac{3}{8}, \quad P(X = 3) = \frac{1}{8}$$

1.1.2 General Properties of the Binomial Distribution

The **expected value** or mean of a binomial random variable is given by:

$$E(X) = n \cdot p$$

The **variance** of a binomial random variable is:

$$\mathrm{Var}(X) = n \cdot p \cdot (1 - p)$$

These properties help describe the central tendency and spread of the distribution.

1.2 Continuous Probability Distributions

For continuous random variables, the probability distribution is represented by a **probability density function** (PDF). The PDF specifies the relative likelihood of the random variable taking on a particular value. The probability that a continuous random variable X falls within an interval $[a, b]$ is given by the integral of the PDF over that interval:

$$P(a \leq X \leq b) = \int_a^b f_X(x)\, dx$$

where $f_X(x)$ is the PDF of the random variable X.

1.2.1 Normal Distribution

The **normal distribution**, also known as the Gaussian distribution, is one of the most important continuous probability distributions in statistics. It describes a continuous random variable that is symmetrically distributed around its mean, with its shape determined by its mean (μ) and standard deviation (σ).

The PDF of the normal distribution is given by:

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

where:

- μ is the mean of the distribution,
- σ is the standard deviation, and
- x is the value of the random variable.

The normal distribution has the following properties:

- The mean, median, and mode are all equal and located at the center of the distribution.
- The distribution is symmetric around the mean.
- Approximately 68% of the data falls within one standard deviation of the mean, and 95% falls within two standard deviations.

Example: Heights of People

Suppose the height of adult males in a certain population follows a normal distribution with a mean of $\mu = 175$ cm and a standard deviation of $\sigma = 10$ cm. The probability that a randomly selected male has a height between 170 cm and 180 cm can be calculated as:

$$P(170 \leq X \leq 180) = \int_{170}^{180} \frac{1}{10\sqrt{2\pi}} \exp\left(-\frac{(x-175)^2}{2 \cdot 10^2}\right) dx$$

This integral gives the area under the curve between 170 cm and 180 cm, which corresponds to the probability of a height falling within this range.

1.2.2 Standard Normal Distribution

The **standard normal distribution** is a special case of the normal distribution where the mean is 0 and the standard deviation is 1. The PDF of the standard normal distribution is:

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right)$$

We can convert any normal distribution into the standard normal distribution using the **z-score**:

$$Z = \frac{X-\mu}{\sigma}$$

where X is a value from the original normal distribution, μ is the mean, and σ is the standard deviation. The z-score measures how many standard deviations a value is from the mean.

1.3 Illustrations: PMF and PDF

Probability Mass Function (PMF) for a Discrete Random Variable

A probability mass function (PMF) gives the probability that a discrete random variable is exactly equal to a specific value. Below is an example of the PMF for a binomial distribution where the number of trials is $n = 3$ and the probability of success for each trial is $p = \frac{1}{2}$.

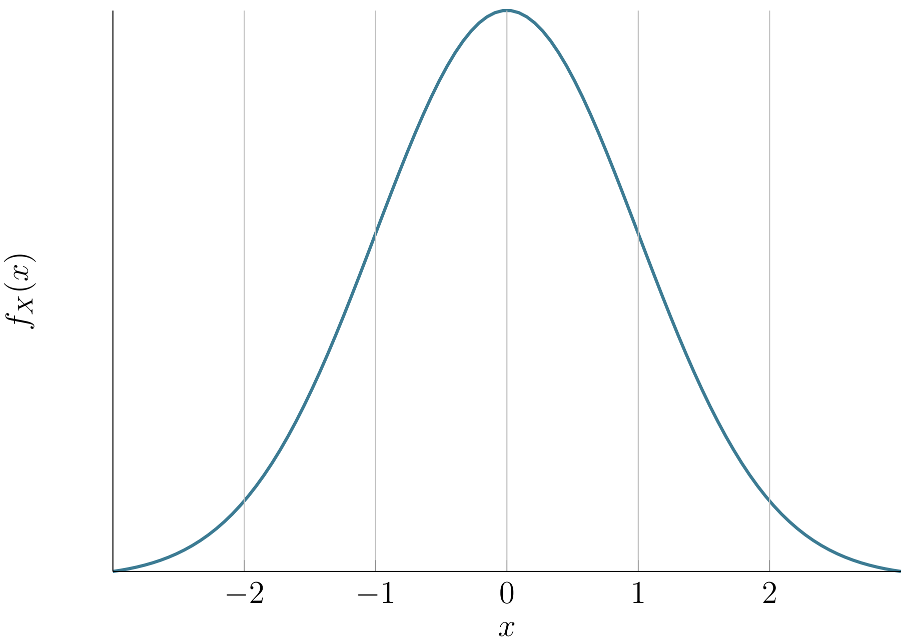
X (Number of Successes)	$P(X)$ (Probability)
0	$\frac{1}{8} = 0.125$
1	$\frac{3}{8} = 0.375$
2	$\frac{3}{8} = 0.375$
3	$\frac{1}{8} = 0.125$

This PMF represents the probabilities of getting 0, 1, 2, or 3 successes in 3 trials, with each trial having a probability of success of $\frac{1}{2}$.

Probability Density Function (PDF) for a Continuous Random Variable

A probability density function (PDF) describes the likelihood of a continuous random variable taking on a specific value. Unlike the PMF, the value of the PDF at any point does not give a probability directly, but the area under the curve over an interval gives the probability that the random variable falls within that interval.

Below is the PDF of the normal distribution with mean $\mu = 0$ and standard deviation $\sigma = 1$, which is also known as the standard normal distribution.



The PDF follows the bell-shaped curve of the normal distribution, which is symmetric around the mean ($\mu = 0$) and falls off as x moves away from the mean. The total area under the curve is 1, representing the entire probability space.

2 Expected Value

The **expected value**, also known as the mean, of a random variable is a measure of the central tendency, representing the long-run average outcome of a random experiment if it were repeated many times. It provides a weighted average of all possible values that a random variable can take, with each value weighted by its probability or probability density.

2.1 Expected Value for Discrete Random Variables

For a discrete random variable X, the expected value is calculated by summing over all possible values of X, each multiplied by its corresponding probability. The formula is:

$$E(X) = \sum_x x \cdot P(X = x)$$

Where: - $E(X)$ is the expected value of the random variable X. - x represents each possible value that X can take. - $P(X = x)$ is the probability that the random variable X takes the value x.

Example:

Consider a fair six-sided die, where each side has an equal probability of $\frac{1}{6}$. The expected value of the roll, denoted X, can be computed as:

$$E(X) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6}$$

$$E(X) = \frac{1+2+3+4+5+6}{6} = \frac{21}{6} = 3.5$$

Thus, the expected value of a roll of a fair die is 3.5.

2.2 Expected Value for Continuous Random Variables

For a continuous random variable X, the expected value is calculated using an integral that sums the product of the value of X and its probability density function (PDF) $f(x)$ over all possible values of X. The formula is:

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$$

Where: - $E(X)$ is the expected value of the continuous random variable X. - x represents the values of X. - $f(x)$ is the probability density function (PDF) of X.

Example:

For a standard normal distribution, where the PDF is given by:

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

The expected value of X is 0, since the normal distribution is symmetric around the mean $\mu = 0$:

$$E(X) = \int_{-\infty}^{\infty} x \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \, dx = 0$$

Thus, the expected value of a standard normal distribution is 0.

In both discrete and continuous cases, the expected value provides insight into the "center" or "balance point" of the distribution, which can help in understanding the long-term behavior of the random variable.