Summary Page

Vector Magnitude:

- **2D Vector:** $\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2}$
- **3D Vector:** $\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$

Dot Product (for 2D vectors): $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2.$

Dot Product (for 3D vectors): $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$

or Cross Product (Simplified): $\mathbf{u} \times \mathbf{v} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}.$

Cross Product (for 3D vectors): $\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2,\ u_3 v_1 - u_1 v_3,\ u_1 v_2 - u_2 v_1).$

Angle Between Two Vectors:

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} \quad \Longrightarrow \quad \theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}\right)$$

Alternate (Inverse Tangent) Method:

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \quad \text{for a vector } (x, y)$$

2D Rotation Matrix:

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Scaling Matrix (2D):

$$S = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$

Shear Matrices (2D):

$$H = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \quad \text{or} \quad H' = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$$

Projection Matrix (onto x-axis):

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Translation via Homogeneous Coordinates:

$$T = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

Other Concepts:

- **Inverse Vector:** For a vector $\mathbf{v}$, its inverse is $-\mathbf{v}$ such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.
- **Odd Function:** A function $f(x)$ is odd if $f(-x) = -f(x)$ for all x.

CM1015 Computational Mathematics - Vectors and Linear Transformations

1 Introduction

This document covers fundamental aspects of algebra, vectors, matrices, and linear transformations. Topics include definitions and properties of vector spaces, vector operations (including dot product, cross product, and magnitude), and their representation via matrices. In addition, we discuss how matrices capture geometric transformations such as rotations, scaling, shear, and projections. Homogeneous coordinates are introduced for handling translations, and we briefly review Gaussian elimination for solving linear systems.

2 Vectors and Vector Spaces

2.1 Definition of a Vector Space

A **vector space** V over a field (typically $\mathbb{R}$ or $\mathbb{C}$) is a set equipped with two operations:

- **Vector Addition:** For any $\mathbf{u}, \mathbf{v} \in V$, the sum $\mathbf{u} + \mathbf{v}$ is in V.
- **Scalar Multiplication:** For any scalar a and vector $\mathbf{v} \in V$, the product $a\mathbf{v}$ is in V.

These operations satisfy the following properties:

1. **Commutativity:** $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
2. **Associativity:** $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.
3. **Identity Element:** There exists a zero vector $\mathbf{0}$ such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$.
4. **Inverse Element:** For every $\mathbf{v}$ there exists an inverse vector $-\mathbf{v}$ such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.
5. **Distributivity (Scalar over Vector Addition):**

$$a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}.$$

6. **Distributivity (Scalar Addition):**

$$(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}.$$

2.2 Examples and Illustrations

Example 1: Let $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ be two vectors in $\mathbb{R}^2$. Their addition is defined by:

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1,\ u_2 + v_2).$$

Illustration: The TikZ diagram below graphically represents vector addition.

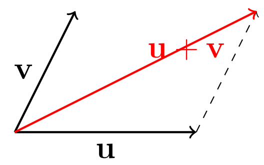


Figure 1: Vector Addition Diagram

2.3 Magnitude and Polar Coordinates

The **magnitude** (or Euclidean norm) of a vector is a measure of its length.

- For a 2D vector $\mathbf{v} = (v_1, v_2)$:
$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2}.$$
- For a 3D vector $\mathbf{v} = (v_1, v_2, v_3)$:
$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}.$$

In polar coordinates, a 2D point is described by a radius r and an angle θ. For a vector $\mathbf{v} = (x, y)$:

$$r = \|\mathbf{v}\|, \quad \theta = \arctan\left(\frac{y}{x}\right).$$

Alternatively, the angle θ between two vectors $\mathbf{u}$ and $\mathbf{v}$ can be computed by:

$$\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}\right).$$

Both methods (inverse tangent and inverse cosine) are useful depending on the information available.

Magnitude Examples:

- For a 2D vector $\mathbf{a} = (3, 4)$:
$$\|\mathbf{a}\| = \sqrt{3^2 + 4^2} = 5.$$
- For a 3D vector $\mathbf{b} = (2, -1, 2)$:
$$\|\mathbf{b}\| = \sqrt{2^2 + (-1)^2 + 2^2} = \sqrt{4 + 1 + 4} = \sqrt{9} = 3.$$

Odd Functions: A function $f(x)$ is **odd** if it satisfies:

$$f(-x) = -f(x) \quad \text{for all } x.$$

For example, $f(x) = x^3$ is odd since $(-x)^3 = -x^3$.

3 Operations with Vectors

3.1 Dot Product (Scalar Product)

The **dot product** of two vectors $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ is defined as:

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

Example: For $\mathbf{u} = (1, 3)$ and $\mathbf{v} = (4, -2)$,

$$\mathbf{u} \cdot \mathbf{v} = 1 \cdot 4 + 3 \cdot (-2) = 4 - 6 = -2.$$

3.2 Cross Product

The **cross product** is defined for vectors in $\mathbb{R}^3$. For $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$:

$$\mathbf{u} \times \mathbf{v} = \Big(u_2 v_3 - u_3 v_2,\ u_3 v_1 - u_1 v_3,\ u_1 v_2 - u_2 v_1 \Big).$$

Example: If $\mathbf{u} = (1, 0, 0)$ and $\mathbf{v} = (0, 1, 0)$, then:

$$\mathbf{u} \times \mathbf{v} = (0 \cdot 0 - 0 \cdot 1,\ 0 \cdot 0 - 1 \cdot 0,\ 1 \cdot 1 - 0 \cdot 0) = (0, 0, 1).$$

4 Matrices and Linear Transformations

4.1 Linear Transformations and Their Matrix Representations

A **linear transformation** $T : \mathbb{R}^n \to \mathbb{R}^m$ satisfies:

$$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \quad \text{and} \quad T(a\mathbf{v}) = a\,T(\mathbf{v}).$$

Any linear transformation can be represented by a matrix A such that:

$$T(\mathbf{x}) = A\mathbf{x}.$$

Example: A rotation in $\mathbb{R}^2$ by an angle θ is represented by:

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

4.2 Geometric Transformations

Matrices can represent various geometric transformations:

- **Scaling:**
$$S = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}.$$
- **Rotation:** (as shown above)
- **Shear:**
$$H = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \quad \text{or} \quad H' = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}.$$
- **Projection:** For projecting onto the x-axis:
$$P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

4.3 Translations and Homogeneous Coordinates

Translations are not linear in standard Cartesian coordinates. To handle translations using matrices, we use **homogeneous coordinates**. For a 2D point (x, y), we augment it to $(x, y, 1)$ and a translation by (t_x, t_y) is given by:

$$T = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus, the translated point is:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = T \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}.$$

4.4 Composition of Transformations

The composition of two linear transformations corresponds to the multiplication of their matrices. If

$$T_1(\mathbf{x}) = A_1\mathbf{x} \quad \text{and} \quad T_2(\mathbf{x}) = A_2\mathbf{x},$$

then the composite transformation $T_2 \circ T_1$ is given by:

$$T_2(T_1(\mathbf{x})) = A_2(A_1\mathbf{x}) = (A_2A_1)\mathbf{x}.$$

Example: Rotate a vector by θ and then scale it by factors s_x and s_y. The combined transformation matrix is:

$$A = S \cdot R(\theta) = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

5 Systems of Linear Equations and Gaussian Elimination

5.1 Gaussian Elimination

Gaussian elimination is a systematic method for solving systems of linear equations. Consider the system:

$$\begin{aligned} x + 2y - z &= 3, \\ 2x - y + 3z &= 4, \\ -x + 4y + 2z &= 5. \end{aligned}$$

The method involves:

1. Writing the augmented matrix for the system.
2. Applying row operations to reduce the matrix to row-echelon form.
3. Back-substituting to obtain the solution.

5.2 Homogeneous Equations

A homogeneous system is one of the form:

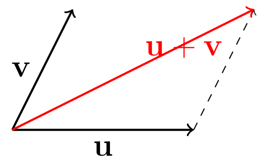
$$A\mathbf{x} = \mathbf{0}.$$

The trivial solution is always $\mathbf{x} = \mathbf{0}$. Non-trivial solutions exist if the determinant of A is zero, indicating the presence of free parameters.

6 Examples and Illustrations

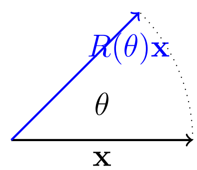
6.1 Vector Addition Diagram (Working Version)

Below is a TikZ illustration for vector addition:



6.2 Rotation Transformation Diagram

Below is an example TikZ diagram illustrating the rotation of a vector:



6.3 Magnitude, Dot Product, and Cross Product Examples

Magnitude Examples:

- For a 2D vector $\mathbf{a} = (3, 4)$:
$$\|\mathbf{a}\| = \sqrt{3^2 + 4^2} = 5.$$
- For a 3D vector $\mathbf{b} = (2, -1, 2)$:
$$\|\mathbf{b}\| = \sqrt{2^2 + (-1)^2 + 2^2} = \sqrt{4 + 1 + 4} = 3.$$

Dot Product Example:

Let $\mathbf{u} = (1, 0)$ and $\mathbf{v} = (1, 1)$. Then,

$$\mathbf{u} \cdot \mathbf{v} = 1 \cdot 1 + 0 \cdot 1 = 1.$$

Computing the magnitudes:

$$\|\mathbf{u}\| = \sqrt{1^2 + 0^2} = 1, \quad \|\mathbf{v}\| = \sqrt{1^2 + 1^2} = \sqrt{2}.$$

Using the inverse cosine method, the angle between $\mathbf{u}$ and $\mathbf{v}$ is:

$$\theta = \cos^{-1}\left(\frac{1}{1 \cdot \sqrt{2}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) \approx 45°.$$

Alternatively, using the inverse tangent method for $\mathbf{v} = (1, 1)$:

$$\theta = \tan^{-1}\left(\frac{1}{1}\right) = \tan^{-1}(1) \approx 45^\circ.$$

Cross Product Example:

For $\mathbf{u} = (1, 0, 0)$ and $\mathbf{v} = (0, 1, 0)$,

$$\mathbf{u} \times \mathbf{v} = \Big(0 \cdot 0 - 0 \cdot 1,\ 0 \cdot 0 - 1 \cdot 0,\ 1 \cdot 1 - 0 \cdot 0\Big) = (0, 0, 1).$$