Self Assignment Vector Spaces (Subspaces, Basis, Dimension)

2024

Vector Spaces

A **vector space** (or linear space) is a collection of objects called vectors, which can be added together and multiplied (scaled) by numbers, called scalars. Scalars are often real numbers ($\mathbb{R}$) but can also come from other fields like complex numbers ($\mathbb{C}$).

Formally, a vector space V over a field F (usually $\mathbb{R}$ or $\mathbb{C}$) satisfies the following properties: - **Closure under addition**: For any vectors $u, v \in V$, $u + v \in V$. - **Closure under scalar multiplication**: For any scalar $c \in F$ and any vector $v \in V$, $cv \in V$. - **Zero vector**: There exists a zero vector $0 \in V$ such that for any vector $v \in V$, $v + 0 = v$. - **Additive inverses**: For each $v \in V$, there is a vector $-v \in V$ such that $v + (-v) = 0$. - **Distributive and associative properties** for vector addition and scalar multiplication.

Example 1: $\mathbb{R}^2$

The set of all 2-dimensional vectors with real components, $\mathbb{R}^2$, is an example of a vector space:

$$\mathbb{R}^2 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x, y \in \mathbb{R} \right\}.$$

Vectors can be added, and they can be multiplied by scalars from $\mathbb{R}$.

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}, \quad 2 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}.$$

Subspaces

A **subspace** of a vector space V is a subset $W \subseteq V$ that is itself a vector space under the same operations of vector addition and scalar multiplication.

For W to be a subspace, it must satisfy: 1. The zero vector of V is in W. 2. W is closed under vector addition. 3. W is closed under scalar multiplication.

Example 2: Subspace of $\mathbb{R}^2$

The set of all vectors on a line through the origin in $\mathbb{R}^2$ is a subspace. For example, the line $y = 2x$ forms the subspace

$$W = \left\{ \begin{pmatrix} x \\ 2x \end{pmatrix} : x \in \mathbb{R} \right\}.$$

This is a subspace because it contains the zero vector $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, and it is closed under both addition and scalar multiplication.

Basis

A **basis** of a vector space V is a set of vectors in V that are linearly independent and span the entire vector space. This means every vector in V can be written as a linear combination of the basis vectors.

If $B = \{v_1, v_2, \ldots, v_n\}$ is a basis for V, then for every vector $v \in V$, there are unique scalars $a_1, a_2, \ldots, a_n$ such that:

$$v = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n.$$

Example 3: Basis of $\mathbb{R}^2$

The standard basis for $\mathbb{R}^2$ is:

$$B = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$$

Any vector in $\mathbb{R}^2$ can be expressed as a linear combination of these two vectors. For example:

$$\begin{pmatrix} 3 \\ 5 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 5 \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Dimension

The **dimension** of a vector space V is the number of vectors in any basis for V. If a vector space V has a basis consisting of n vectors, we say that V is n-dimensional, and we write:

$$\dim(V) = n.$$

Example 4: Dimension of $\mathbb{R}^3$

The standard basis for $\mathbb{R}^3$ consists of three vectors:

$$B = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Thus, the dimension of $\mathbb{R}^3$ is 3.

$$\dim(\mathbb{R}^3) = 3.$$

Linear Independence

A set of vectors $\{v_1, v_2, \ldots, v_n\}$ in a vector space V is called **linearly independent** if the only solution to the equation:

$$a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = 0$$

is $a_1 = a_2 = \cdots = a_n = 0$. In other words, no vector in the set can be written as a linear combination of the others.

Example 5: Linearly Independent Vectors in $\mathbb{R}^2$

The vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are linearly independent because the only solution to:

$$a_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

is $a_1 = 0$ and $a_2 = 0$.

Diagram: Visualizing a Subspace in $\mathbb{R}^2$

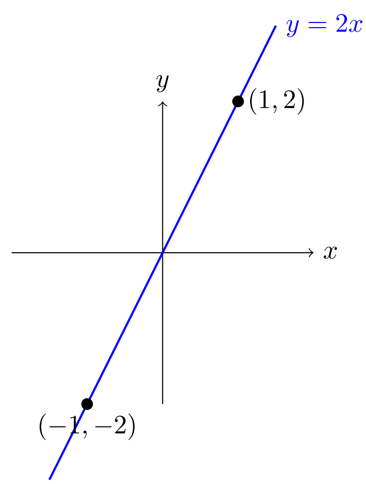

Figure 1: A subspace of $\mathbb{R}^2$ represented by the line $y = 2x$. This subspace contains all scalar multiples of the vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.