CM1015 - Matrix Theory

1 Example of a Matrix

Consider the matrices

$$E = \begin{bmatrix} 2 & 0 & -1 \\ 3 & 1 & 4 \\ -2 & 5 & 0 \end{bmatrix} \quad \text{and} \quad F = \begin{bmatrix} 1 & 3 & 2 \\ -1 & 0 & 5 \\ 4 & -2 & 1 \end{bmatrix}.$$

We will perform three operations: matrix addition, scalar multiplication, and matrix multiplication.

1.1 Matrix Addition

To add E and F, add the corresponding entries:

$$E + F = \begin{bmatrix} 2+1 & 0+3 & -1+2 \\ 3+(-1) & 1+0 & 4+5 \\ -2+4 & 5+(-2) & 0+1 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 1 \\ 2 & 1 & 9 \\ 2 & 3 & 1 \end{bmatrix}.$$

Each element in the resulting matrix is the sum of the elements in the same position from E and F.

1.2 Scalar Multiplication

Multiply the matrix E by a scalar, for instance 2:

$$2E = \begin{bmatrix} 2\cdot 2 & 2\cdot 0 & 2\cdot(-1) \\ 2\cdot 3 & 2\cdot 1 & 2\cdot 4 \\ 2\cdot(-2) & 2\cdot 5 & 2\cdot 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 & -2 \\ 6 & 2 & 8 \\ -4 & 10 & 0 \end{bmatrix}.$$

This operation scales every entry of E by 2.

1.3 Matrix Multiplication

The product EF is found by taking the dot product of rows of E with columns of F. In general,

$$(EF)_{ij} = \sum_{k=1}^{3} e_{ik} f_{kj}.$$

We compute each entry step by step:

- $(EF)_{11} = 2 \cdot 1 + 0 \cdot (-1) + (-1) \cdot 4 = 2 + 0 - 4 = -2$.
- $(EF)_{12} = 2 \cdot 3 + 0 \cdot 0 + (-1) \cdot (-2) = 6 + 0 + 2 = 8$.
- $(EF)_{13} = 2 \cdot 2 + 0 \cdot 5 + (-1) \cdot 1 = 4 + 0 - 1 = 3$.
- $(EF)_{21} = 3 \cdot 1 + 1 \cdot (-1) + 4 \cdot 4 = 3 - 1 + 16 = 18$.
- $(EF)_{22} = 3 \cdot 3 + 1 \cdot 0 + 4 \cdot (-2) = 9 + 0 - 8 = 1$.
- $(EF)_{23} = 3 \cdot 2 + 1 \cdot 5 + 4 \cdot 1 = 6 + 5 + 4 = 15$.
- $(EF)_{31} = (-2) \cdot 1 + 5 \cdot (-1) + 0 \cdot 4 = -2 - 5 + 0 = -7$.
- $(EF)_{32} = (-2) \cdot 3 + 5 \cdot 0 + 0 \cdot (-2) = -6 + 0 + 0 = -6$.
- $(EF)_{33} = (-2) \cdot 2 + 5 \cdot 5 + 0 \cdot 1 = -4 + 25 + 0 = 21$.

Thus, the product is

$$EF = \begin{bmatrix} -2 & 8 & 3 \\ 18 & 1 & 15 \\ -7 & -6 & 21 \end{bmatrix}.$$

1.4 Illustration and Explanation

Imagine each row of matrix E as a recipe and each column of matrix F as a list of ingredients. To calculate an entry in the product EF:

1. Multiply each number in a row of E (the recipe) by the corresponding number in a column of F (the ingredients).
2. Sum these products to get one entry of the new matrix.

This method, called the dot product, is applied for every combination of rows and columns, producing the complete matrix EF.

This additional example shows how matrix operations can be applied to larger, more complex matrices while keeping the process straightforward.

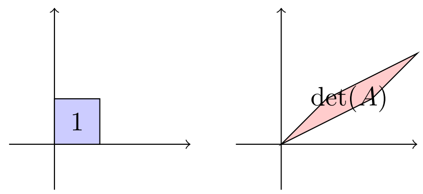
2 Determinant and Inverse

2.1 Determinant

The determinant of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is:

$$\det(A) = ad - bc$$

Geometrically, it represents the area scaling factor of the linear transformation described by A.



2.2 Inverse of a 2×2 Matrix

If $\det(A) \neq 0$, the inverse is:

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example: For $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, $\det(A) = 1$, so:

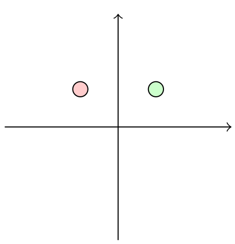
$$A^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

3 Transformations with 2×2 Matrices

3.1 Reflections

Reflection matrices flip points over a line. They are self-inverse ($R^2 = I$):

$$\text{y-axis: } \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{y=x: } \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$



3.2 Rotations

Rotation by θ degrees:

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Example (90°):

$$R(90^\circ) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

3.3 Additional Rotation Examples

Below are two more examples of rotations using the rotation matrix. Recall that a rotation by an angle θ is given by:

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

3.3.1 Example 1: Rotation by 45°

Step 1: Write the Rotation Matrix. For $\theta = 45°$, we know that

$$\cos 45° = \sin 45° = \frac{\sqrt{2}}{2}.$$

Thus, the rotation matrix becomes:

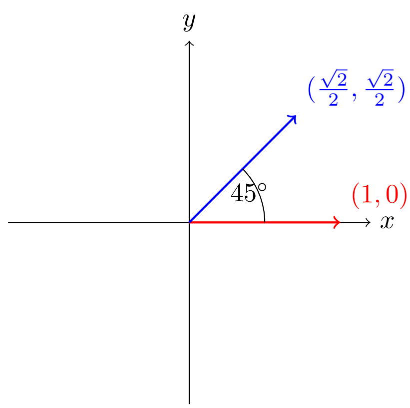
$$R(45°) = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}.$$

Step 2: Apply the Rotation to a Point. Consider the point $P = (1, 0)$. To find its new coordinates P' after a 45° rotation, multiply:

$$P' = R(45°)\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \cdot 1 + \left(-\frac{\sqrt{2}}{2}\right) \cdot 0 \\ \frac{\sqrt{2}}{2} \cdot 1 + \frac{\sqrt{2}}{2} \cdot 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}.$$

This shows that the point $(1, 0)$ moves to $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

Step 3: Illustration. Imagine the coordinate plane where the original point P lies on the positive x-axis. After rotation by 45°, P' lies in the first quadrant at a 45° angle from the x-axis.



3.3.2 Example 2: Rotation by 180°

Step 1: Write the Rotation Matrix. For $\theta = 180^\circ$, recall that:

$$\cos 180^\circ = -1 \quad \text{and} \quad \sin 180^\circ = 0.$$

Thus, the rotation matrix is:

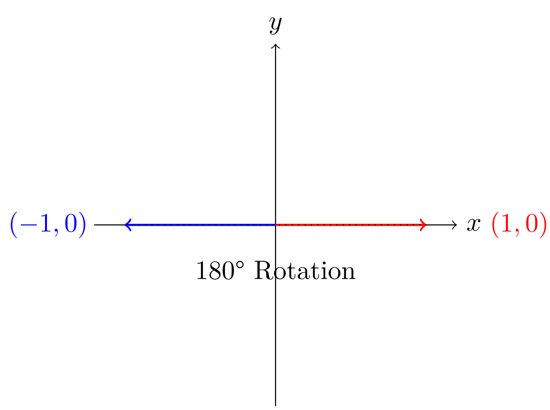
$$R(180^\circ) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Step 2: Apply the Rotation to a Point. Using the same point $P = (1, 0)$, we compute:

$$P' = R(180^\circ) \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \cdot 1 + 0 \cdot 0 \\ 0 \cdot 1 + (-1) \cdot 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

The point $(1, 0)$ is rotated to $(-1, 0)$, meaning it is reflected through the origin.

Step 3: Illustration. On the coordinate plane, the original point P lies on the positive x-axis. After a 180° rotation, it appears on the negative x-axis.



4 Shearing Transformation

Shearing is a transformation that shifts points in a specific direction by an amount proportional to their coordinate values. It distorts the shape by slanting it while preserving parallel lines.

4.1 Shear Matrices

In two dimensions, there are two basic types of shear transformations:

- **X-shear:** Moves each point (x, y) to $(x + k\,y, y)$. Its transformation matrix is:

$$S_x(k) = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}.$$

- **Y-shear:** Moves each point (x, y) to $(x, y+k\,x)$. Its transformation matrix is:
$$S_y(k) = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}.$$

4.2 Step-by-Step Example: X-Shear

Consider a quadrilateral with the following vertices:

$$P_1 = (1, 1), \quad P_2 = (4, 1), \quad P_3 = (5, 3), \quad P_4 = (2, 3).$$

We apply an x-shear with a shear factor $k = 1.5$. This means each point (x, y) is transformed into:

$$x' = x + 1.5\,y, \quad y' = y.$$

4.2.1 Applying the Shear Transformation

1. Transform $P_1 = (1, 1)$:
$$x' = 1 + 1.5 \times 1 = 2.5, \quad y' = 1.$$
New point: $P_1' = (2.5, 1)$.

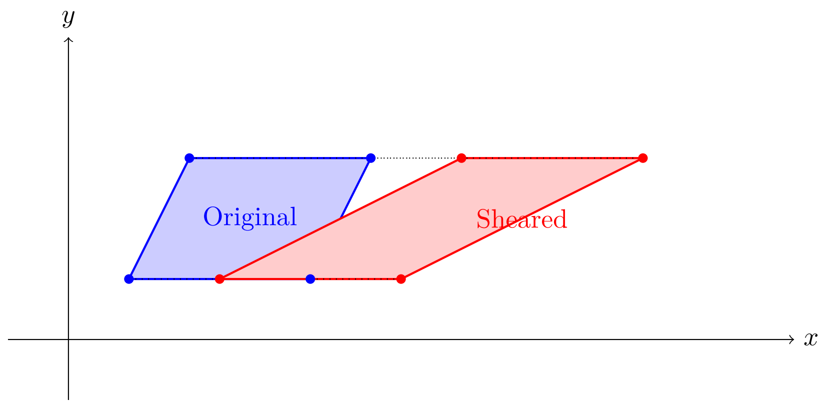
2. Transform $P_2 = (4, 1)$:
$$x' = 4 + 1.5 \times 1 = 5.5, \quad y' = 1.$$
New point: $P_2' = (5.5, 1)$.

3. Transform $P_3 = (5, 3)$:
$$x' = 5 + 1.5 \times 3 = 9.5, \quad y' = 3.$$
New point: $P_3' = (9.5, 3)$.

4. Transform $P_4 = (2, 3)$:
$$x' = 2 + 1.5 \times 3 = 6.5, \quad y' = 3.$$
New point: $P_4' = (6.5, 3)$.

4.2.2 Illustration of Shearing

The following TikZ diagram shows the original quadrilateral in blue and the transformed (sheared) quadrilateral in red.



4.3 Shear Properties and Observations

- The transformation does not affect the y-coordinates.
- Vertical lines become slanted.
- Parallel lines remain parallel after shearing.
- The overall area of the shape is preserved, but angles between sides are changed.

4.4 Example of Y-Shear

Now, let's consider the same quadrilateral and apply a y-shear with $k = 1.2$, meaning each point transforms as:

$$x' = x, \quad y' = y + 1.2\,x.$$

Transform each vertex:

1. $P_1 = (1, 1) \Rightarrow P_1' = (1, 1 + 1.2 \times 1) = (1, 2.2)$.
2. $P_2 = (4, 1) \Rightarrow P_2' = (4, 1 + 1.2 \times 4) = (4, 5.8)$.
3. $P_3 = (5, 3) \Rightarrow P_3' = (5, 3 + 1.2 \times 5) = (5, 9)$.
4. $P_4 = (2, 3) \Rightarrow P_4' = (2, 3 + 1.2 \times 2) = (2, 5.4)$.

Applying the y-shear skews the shape in the vertical direction.

5 Odd and Even Functions in Rotation Matrices

A function f is called *even* if $f(-x) = f(x)$ for every x, and *odd* if $f(-x) = -f(x)$ for every x. In trigonometry, these properties are shown by:

$$\cos(-\theta) = \cos\theta \quad \text{(even)} \quad \text{and} \quad \sin(-\theta) = -\sin\theta \quad \text{(odd)}.$$

The rotation matrix for an angle θ is given by:

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

If we want to rotate by the negative angle $-\theta$, we have:

$$R(-\theta) = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

Notice that this matrix is exactly the transpose of $R(\theta)$:

$$R(\theta)^\top = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

Thus, we have:

$$R(-\theta) = R(\theta)^\top.$$

Since orthogonal matrices satisfy $R(\theta)^\top R(\theta) = I$, it follows that

$$R(\theta)R(-\theta) = R(\theta)R(\theta)^\top = I,$$

which shows that rotating by θ and then by $-\theta$ returns a vector to its original position.

Example: Rotation by 30°

Let $\theta = 30^\circ$. We know:

$$\cos 30^\circ = \frac{\sqrt{3}}{2}, \quad \sin 30^\circ = \frac{1}{2}.$$

Then, the rotation matrix is:

$$R(30^\circ) = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix},$$

and the matrix for -30° is:

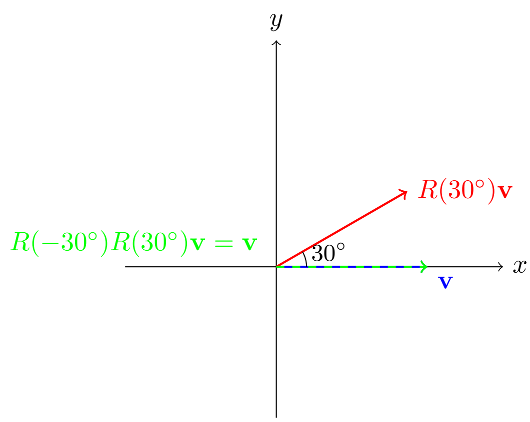
$$R(-30^\circ) = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}.$$

Notice that $R(-30^\circ) = R(30^\circ)^\top$. Multiplying the two matrices, we obtain:

$$R(30^\circ)R(-30^\circ) = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

Illustration

Below is a TikZ illustration demonstrating the effect of rotating a vector by θ and then by $-\theta$:



Summary

- The cosine function is even: $\cos(-\theta) = \cos\theta$.
- The sine function is odd: $\sin(-\theta) = -\sin\theta$.
- Thus, the rotation matrix for $-\theta$ is the transpose of the rotation matrix for θ:
$$R(-\theta) = R(\theta)^\top.$$
- Consequently, $R(\theta)R(-\theta) = I$, which confirms that rotation matrices are orthogonal.