Self Assignment: Matrix Decomposition's (LU, QR, SVD)

2024

1 Matrix Decomposition's

Matrix decomposition's are techniques used to factor matrices into simpler forms. This process is incredibly useful for solving linear equations, performing numerical analysis, and in machine learning algorithms. Understanding these decomposition's is essential for advanced courses in computer science, particularly in fields such as data science, numerical methods, optimization, and computer vision.

In this document, we will cover three essential matrix decomposition's:

- LU Decomposition
- QR Decomposition
- Singular Value Decomposition (SVD)

We will explain each decomposition step-by-step, along with examples, applications, and diagrams to give you a strong foundation.

1.1 LU Decomposition

LU decomposition factors a matrix A into two triangular matrices:

$$A = LU$$

where: - L is a lower triangular matrix (with ones on the diagonal). - U is an upper triangular matrix.

Why is LU Decomposition useful? LU decomposition is useful for solving systems of linear equations, particularly when the same matrix is used with different right-hand sides. It is also used in calculating matrix determinants and inverses, and in numerical stability analyses.

1.1.1 Detailed Explanation

Given an $n \times n$ matrix A, LU decomposition expresses A as the product of two matrices L and U, where:

$$L = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ l_{21} & 1 & 0 & \dots & 0 \\ l_{31} & l_{32} & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ l_{n1} & l_{n2} & l_{n3} & \dots & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} & \dots & u_{1n} \\ 0 & u_{22} & u_{23} & \dots & u_{2n} \\ 0 & 0 & u_{33} & \dots & u_{3n} \\ \vdots & \vdots & \vdots & \ddots & u_{nn} \\ 0 & 0 & 0 & \dots & u_{nn} \end{bmatrix}$$

1.1.2 Example of LU Decomposition

Let's break down a 3x3 matrix using LU decomposition.

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 7 & -1 \\ -2 & 4 & 5 \end{bmatrix}$$

Step 1: Apply Gaussian elimination to reduce A into upper triangular form U:

$$U = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 2 \end{bmatrix}$$

Step 2: Construct the lower triangular matrix L by recording the factors used to eliminate the lower elements:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 4 & 1 \end{bmatrix}$$

Thus, $A = LU$, where:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 4 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 2 \end{bmatrix}$$

1.1.3 Applications of LU Decomposition

1. **Solving Linear Systems:** Once a matrix A is decomposed into LU, solving $Ax = b$ becomes easier by solving $Ly = b$ and then $Ux = y$ through forward and backward substitution. 2. **Matrix Inversion:** Inverting A can be performed by inverting L and U, which is computationally simpler. 3. **Determinants:** The determinant of A can be easily computed as the product of the diagonal elements of U.

1.2 QR Decomposition

QR decomposition breaks a matrix A into:

$$A = QR$$

where: - Q is an orthogonal matrix (i.e., $Q^T Q = I$). - R is an upper triangular matrix.

Why is QR Decomposition useful? QR decomposition is used in solving linear systems, least squares problems, and in algorithms for eigenvalue computation (like the QR algorithm).

1.2.1 Detailed Explanation

Given an $m \times n$ matrix A, the QR decomposition represents A as the product of an orthogonal matrix Q (with orthonormal columns) and an upper triangular matrix R. The orthonormality of Q means that the columns of Q are perpendicular to each other and have unit length.

1.2.2 Example of QR Decomposition

Consider the matrix:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Step 1: We apply the Gram-Schmidt process to orthogonalize the columns of A. The result is:

$$Q = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

Step 2: Compute R by multiplying Q^T with A:

$$R = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 2 \end{bmatrix}$$

Thus, $A = QR$.

1.2.3 Applications of QR Decomposition

1. **Solving Least Squares Problems:** QR decomposition is used to solve $Ax = b$ when A is not square by minimizing the residual $\|Ax - b\|$. 2. **Eigenvalue Computation:** The QR algorithm is a common method to compute the eigenvalues and eigenvectors of a matrix. 3. **Stability:** QR decomposition provides a numerically stable method for matrix factorization, especially for ill-conditioned matrices.

1.3 Singular Value Decomposition (SVD)

SVD is one of the most powerful matrix factorizations and represents a matrix A as:

$$A = U\Sigma V^T$$

where: - U is an orthogonal matrix containing the left singular vectors. - Σ is a diagonal matrix with the singular values. - V^T is the transpose of an orthogonal matrix containing the right singular vectors.

Why is SVD important? SVD is fundamental in areas like dimensionality reduction, data compression, and machine learning, particularly in principal component analysis (PCA) and latent semantic analysis (LSA).

1.3.1 Detailed Explanation

The matrix A is decomposed into three components: - U: Columns of U are the eigenvectors of AA^T. - Σ: The diagonal elements of Σ are the square roots of the eigenvalues of A^TA, also known as the singular values. - V: Columns of V are the eigenvectors of A^TA.

1.3.2 Example of SVD

Let's consider the matrix:

$$A = \begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix}$$

Step 1: Compute A^TA and AA^T, then find their eigenvalues and eigenvectors.

Step 2: The singular values are the square roots of the eigenvalues of A^TA.

For this matrix, we obtain:

$$U = \begin{bmatrix} 0.8 & 0.6 \\ 0.6 & -0.8 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}, \quad V^T = \begin{bmatrix} 0.6 & 0.8 \\ -0.8 & 0.6 \end{bmatrix}$$

Thus, $A = U\Sigma V^T$.

1.3.3 Applications of SVD

1. **Dimensionality Reduction:** In machine learning and data science, SVD is used to reduce the dimensionality of data while preserving important information (e.g., PCA). 2. **Image Compression:** SVD can be used to approximate an image matrix with fewer singular values, leading to significant compression. 3. **Latent Semantic Analysis (LSA):** In natural language processing, SVD is used to discover relationships between terms and documents in large text corpora.

2 Conclusion

Matrix decompositions such as LU, QR, and SVD play a critical role in various computational fields, including numerical linear algebra, machine learning, and optimization. Mastery of these concepts is crucial for students at Stanford and beyond, as they are the foundation for many advanced algorithms and applications.