Self Assignment: Inner Product spaces and Diagonalization

2024

1 Inner Product Spaces

1.1 Definition of Inner Product Spaces

An **inner product space** is a vector space V over the field $\mathbb{R}$ (real numbers) or $\mathbb{C}$ (complex numbers), equipped with a binary operation called the inner product. This operation takes two vectors $\mathbf{u}, \mathbf{v} \in V$ and returns a scalar $\langle \mathbf{u}, \mathbf{v} \rangle$. The inner product generalizes the dot product and allows us to measure angles, lengths, and define geometric properties in higher dimensions.

For an inner product, the following properties hold:

1. Linearity in the first argument:

$$\langle a\mathbf{u} + b\mathbf{v}, \mathbf{w} \rangle = a\langle \mathbf{u}, \mathbf{w} \rangle + b\langle \mathbf{v}, \mathbf{w} \rangle$$

2. Conjugate symmetry (for complex spaces):

$$\langle \mathbf{u}, \mathbf{v} \rangle = \overline{\langle \mathbf{v}, \mathbf{u} \rangle}$$

3. Positivity:

$$\langle \mathbf{u}, \mathbf{u} \rangle \geq 0 \quad \text{and} \quad \langle \mathbf{u}, \mathbf{u} \rangle = 0 \quad \text{if and only if} \quad \mathbf{u} = \mathbf{0}$$

1.2 Norms and Inner Products

A **norm** is a function that assigns a length to vectors. For any vector $\mathbf{u} \in V$, its norm is defined using the inner product:

$$\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle}$$

This norm tells us the "magnitude" or length of the vector $\mathbf{u}$.

Properties of norms: - $\|\mathbf{u}\| \geq 0$ (non-negativity), - $\|\mathbf{u}\| = 0$ if and only if $\mathbf{u} = 0$, - $\|c\mathbf{u}\| = |c|\|\mathbf{u}\|$ for any scalar c, - The triangle inequality: $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$.

1.3 Examples of Inner Product Spaces

1. **Real space** $\mathbb{R}^n$:
   - The standard inner product (dot product) in $\mathbb{R}^n$ for vectors $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ is:
   $$\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{i=1}^{n} u_i v_i$$
   - This measures the alignment of vectors.
2. **Complex space** $\mathbb{C}^n$:
   - For complex vectors $\mathbf{u}$ and $\mathbf{v}$, the inner product is:
   $$\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{i=1}^{n} u_i \overline{v_i}$$
   - where $\overline{v_i}$ denotes the complex conjugate of v_i.

1.4 Orthogonality and Orthonormal Sets

Orthogonality: Two vectors $\mathbf{u}$ and $\mathbf{v}$ in a vector space V are said to be **orthogonal** if their inner product is zero:

$$\langle \mathbf{u}, \mathbf{v} \rangle = 0$$

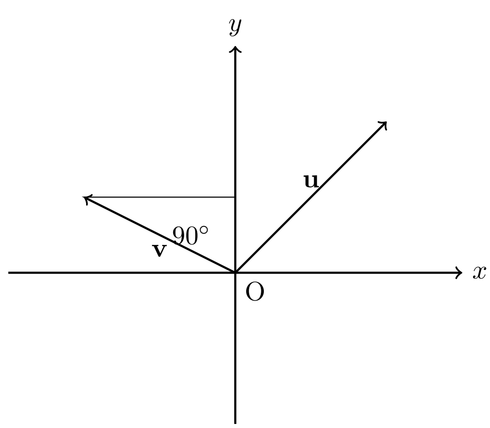
In a geometric sense, orthogonal vectors are perpendicular to each other. This orthogonality simplifies many linear algebra operations and is a key concept in projections and decompositions.

Example: Consider two vectors $\mathbf{u} = (1, 2)$ and $\mathbf{v} = (-2, 1)$ in $\mathbb{R}^2$. Their inner product is:

$$\langle \mathbf{u}, \mathbf{v} \rangle = (1)(-2) + (2)(1) = -2 + 2 = 0$$

Since the inner product is zero, the vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

Diagram of Orthogonality:



Orthonormal Sets: An **orthonormal set** is a set of vectors where each pair of vectors is orthogonal, and each vector has a unit norm:

$$\|\mathbf{v}\| = 1$$

Mathematically, a set of vectors $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_k\}$ is orthonormal if:

$$\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \delta_{ij}$$

where δ_{ij} is the Kronecker delta, which is 1 if $i = j$ and 0 otherwise.

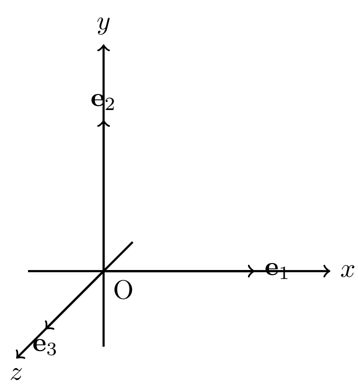
Example: In $\mathbb{R}^3$, the standard basis vectors $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, and $\mathbf{e}_3 = (0, 0, 1)$ form an orthonormal set. We have:

$$\langle \mathbf{e}_1, \mathbf{e}_2 \rangle = 0, \quad \langle \mathbf{e}_1, \mathbf{e}_3 \rangle = 0, \quad \langle \mathbf{e}_2, \mathbf{e}_3 \rangle = 0$$

and

$$\|\mathbf{e}_1\| = 1, \quad \|\mathbf{e}_2\| = 1, \quad \|\mathbf{e}_3\| = 1$$

Diagram of Orthonormal Set:



Importance: Orthonormal sets are particularly useful in simplifying calculations, especially in methods like the Gram-Schmidt process for orthogonalizing vectors. They also form the basis for Fourier series and other decompositions in signal processing and various applications in computer science and engineering.

1.5 Applications of Inner Products

Inner products are fundamental in various fields, providing a way to quantify relationships and interactions between vectors. Here, we explore some of their key applications:

Machine Learning: Inner products are crucial in machine learning for measuring similarities and projections between data points. For instance:

- **Support Vector Machines (SVMs):** In SVMs, the inner product is used to compute the dot product between feature vectors. This helps in determining the decision boundary that separates different classes in high-dimensional spaces. The kernel trick, which uses inner products in feature space, enables SVMs to perform classification in non-linear decision boundaries by implicitly mapping data into higher-dimensional space.

- **Cosine Similarity:** This measure of similarity between two non-zero vectors $\mathbf{u}$ and $\mathbf{v}$ is defined as:

$$\text{Cosine Similarity} = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\| \|\mathbf{v}\|}$$

It quantifies the cosine of the angle between the vectors, providing a measure of similarity that is invariant to the magnitude of the vectors.

Physics: In physics, inner products are used to represent physical quantities and interactions:

- **Work Done:** When a force $\mathbf{F}$ is applied to an object and displaces it by a vector $\mathbf{d}$, the work done W is given by the inner product of these two vectors:

$$W = \langle \mathbf{F}, \mathbf{d} \rangle = \|\mathbf{F}\| \|\mathbf{d}\| \cos \theta$$

where θ is the angle between the force and displacement vectors. This formula quantifies how much of the force contributes to the displacement in the direction of the force.

- **Quantum Mechanics:** In quantum mechanics, inner products are used to compute probabilities and expectation values. For example, the inner product of a state vector with itself gives the probability amplitude, and the inner product of two different state vectors represents the overlap between quantum states.

Additional Applications:

- **Signal Processing:** Inner products are used to analyze and filter signals. Techniques such as Fourier transforms decompose signals into orthogonal components, making it easier to process and analyze them.

- **Computer Graphics:** In graphics, inner products are used to calculate angles and lighting effects. For instance, the dot product of a surface normal vector and a light direction vector determines how much light is hitting the surface, affecting the shading and rendering of objects.

Summary: Inner products are versatile tools that find applications across various domains, from measuring similarities in machine learning to calculating physical quantities in physics. They provide a robust framework for understanding relationships between vectors and performing complex computations efficiently.

2 Operators on Hilbert Spaces

2.1 Linear Operators

A **linear operator** T on a Hilbert space H is a special kind of function that maps vectors in H to other vectors in H. It has two key properties:

- **Additivity:** If you take two vectors $\mathbf{u}$ and $\mathbf{v}$ in H, then:

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$$

  This means that T distributes over addition.

- **Homogeneity:** If you scale a vector $\mathbf{u}$ by a scalar a, then:

$$T(a\mathbf{u}) = aT(\mathbf{u})$$

  This means that T scales with the vector.

- **Example:** Consider a Hilbert space H of all vectors in $\mathbb{R}^2$, and let T be a function defined as:

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 2x \\ 3y \end{pmatrix}$$

  This is a linear operator because it satisfies both additivity and homogeneity.

2.2 Spectral Theorem

The **spectral theorem** is an important result about linear operators, especially in quantum mechanics and signal processing. It states:

- **Self-Adjoint Operators:** If T is a self-adjoint operator (meaning $T = T^*$, where T^* is the adjoint of T), then it can be diagonalized. This means that there exists an orthonormal basis of H consisting of eigenvectors of T, and T can be represented as a diagonal matrix in this basis.

- **Example:** Suppose T is a matrix representing a linear operator in $\mathbb{R}^2$:

$$T = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$$

  The matrix T is symmetric (self-adjoint), and according to the spectral theorem, it can be diagonalized. The eigenvalues and eigenvectors provide the diagonal form of T.

- **Diagram:** In a geometric sense, diagonalization can be visualized as transforming the space so that the operator T simply scales vectors along coordinate axes, rather than rotating or shearing them.

2.3 Positivity of Operators

An operator T is said to be **positive** if:

$$\langle T\mathbf{u}, \mathbf{u} \rangle \geq 0$$

for all vectors $\mathbf{u}$ in H. This means that applying T to any vector $\mathbf{u}$ will result in a vector that has a non-negative projection onto $\mathbf{u}$.

- **Example:** Consider a matrix T in $\mathbb{R}^2$:

  $$T = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

  For any vector $\mathbf{u} = \begin{pmatrix} x \\ y \end{pmatrix}$,

  $$\langle T\mathbf{u}, \mathbf{u} \rangle = \begin{pmatrix} 2x \\ 2y \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = 2x^2 + 2y^2 \geq 0$$

  Hence, T is a positive operator.

- **Applications:** Positive operators are used in quantum mechanics to represent observable quantities, and in optimization problems to ensure that certain criteria are met.

Summary:

- **Linear Operators:** Functions that preserve vector addition and scalar multiplication.
- **Spectral Theorem:** Self-adjoin operators can be diagonalized, simplifying many problems.
- **Positivity:** An operator is positive if it always yields non-negative results when applied to vectors.

This section covers the essentials of operators on Hilbert spaces, providing a foundation for more advanced topics in functional analysis and applications in various scientific fields.

3 Diagonalization

3.1 Eigenvalues and Eigenvectors

In linear algebra, **eigenvalues** and **eigenvectors** are fundamental concepts that help us understand how matrices transform space.

- An **eigenvalue** λ of a square matrix A is a special scalar such that when the matrix A acts on some vector $\mathbf{v}$, the result is simply the eigenvector $\mathbf{v}$ scaled by λ. Mathematically:

$$A\mathbf{v} = \lambda\mathbf{v}$$

  Here, $\mathbf{v}$ is the **eigenvector** corresponding to the eigenvalue λ.

- To find eigenvalues, solve the **characteristic equation**:

$$\det(A - \lambda I) = 0$$

  where I is the identity matrix of the same size as A, and det denotes the determinant of a matrix.

3.2 Example: Finding Eigenvalues and Eigenvectors

Let's find the eigenvalues and eigenvectors of the matrix:

$$A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$$

1. Compute the characteristic equation:

$$\det(A - \lambda I) = \det \begin{bmatrix} 4-\lambda & 1 \\ 2 & 3-\lambda \end{bmatrix}$$

This simplifies to:

$$(4-\lambda)(3-\lambda) - 2 = \lambda^2 - 7\lambda + 10 = 0$$

2. Solve for λ:

$$\lambda^2 - 7\lambda + 10 = 0$$

Factoring or using the quadratic formula gives the eigenvalues:

$$\lambda_1 = 5 \quad \text{and} \quad \lambda_2 = 2$$

3. Find the eigenvectors by substituting each eigenvalue λ back into the equation $(A - \lambda I)\mathbf{v} = 0$.

For $\lambda_1 = 5$:

$$\begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix} \mathbf{v} = \mathbf{0}$$

Solving this gives the eigenvector:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

For $\lambda_2 = 2$:

$$\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \mathbf{v} = \mathbf{0}$$

Solving this gives the eigenvector:

$$\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

3.3 Diagonalization Process

A matrix A is **diagonalizable** if it can be written in the form:

$$A = PDP^{-1}$$

where P is an invertible matrix and D is a diagonal matrix.

- **Diagonal Matrix D:** Contains the eigenvalues of A on its diagonal.
- **Matrix P:** Contains the corresponding eigenvectors of A as its columns.

Example: Using the eigenvalues $\lambda_1 = 5$ and $\lambda_2 = 2$ and their eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$, we can form:

$$P = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}$$

Then:

$$A = PDP^{-1}$$

3.4 Conditions for Diagonalizability

A matrix A is diagonalizable if and only if there are enough linearly independent eigenvectors to form a basis for the space. Specifically, for an $n \times n$ matrix, if there are n linearly independent eigenvectors, the matrix is diagonalizable.

4 Jordan Normal Form

4.1 Jordan Blocks

When a matrix is not diagonalizable, it can be transformed into a **Jordan normal form**. This form is nearly diagonal but consists of **Jordan blocks**. Each Jordan block corresponds to an eigenvalue and has the following structure:

$$J = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & \lambda \end{bmatrix}$$

Here, λ is an eigenvalue, and the 1s on the superdiagonal indicate the presence of generalized eigenvectors.

4.2 Applications of Jordan Normal Form

Jordan normal form is useful in various fields:

- **Differential Equations:** Helps in solving systems of linear differential equations by simplifying the matrix representations.
- **Control Theory:** Used to analyze and design control systems by simplifying the state-space representation of systems.
- **Stability Analysis:** Assists in understanding the stability of dynamical systems by analyzing their Jordan form.

Example: Consider a matrix that is not diagonalizable but can be put into Jordan form. If the matrix A is:

$$A = \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$$

It can be transformed into Jordan form:

$$J = \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$$

where $\lambda = 4$ is the eigenvalue, and the Jordan block captures the non-diagonalizable nature of A.