Self Assignment: Eigenvalues and Eigenvectors

2024

1 Introduction

Eigenvalues and eigenvectors are crucial concepts in linear algebra, widely used across mathematics, physics, engineering, and especially in computer science. Whether it's for simplifying complex transformations, optimizing algorithms, or analyzing large data sets, understanding these concepts is essential.

In layman's terms, an **eigenvector** of a matrix is a direction that remains unchanged when a matrix is applied to it. The **eigenvalue** associated with this eigenvector tells us by how much the vector is scaled—whether it is stretched, compressed, or flipped. This document will break down these concepts, explain them thoroughly, and provide real-world examples where they are used in computer science.

We will:

- Define eigenvalues and eigenvectors in simple terms.
- Provide step-by-step guidance on how to compute them.
- Explain their applications in various fields.
- Use diagrams and examples to help visualize the concepts.

2 What are Eigenvectors and Eigenvalues?

2.1 Eigenvectors

Let's start with the idea of an eigenvector. Imagine applying a transformation to a vector in a 2D or 3D space, like rotating or scaling it. Most vectors will change direction, but some vectors will only be stretched or squished—they keep pointing in the same direction. These special vectors are called **eigenvectors**.

Mathematically, an eigenvector $\mathbf{v}$ of a square matrix A satisfies the following equation:

$$A\mathbf{v} = \lambda\mathbf{v}$$

Here:

- A is a matrix.

- $\mathbf{v}$ is the eigenvector.
- λ is the **eigenvalue**, a scalar.

In simple terms: An eigenvector $\mathbf{v}$ doesn't change direction when multiplied by a matrix A, and the eigenvalue λ tells us how much the eigenvector is scaled. For example, if $\lambda = 2$, the eigenvector is stretched by a factor of 2. If $\lambda = -1$, the vector is flipped but remains on the same line.

2.2 Eigenvalues

The **eigenvalue** λ represents the factor by which the eigenvector is scaled. Eigenvalues can be positive or negative, indicating whether the vector is stretched or flipped.

For instance:

- $\lambda = 1$ means the eigenvector stays the same length.
- $\lambda > 1$ means the eigenvector is stretched.
- $\lambda < 1$ (but positive) means the eigenvector is compressed.
- $\lambda = -1$ means the vector is flipped in the opposite direction.
- $\lambda = 0$ means the vector collapses to zero.

3 How to Find Eigenvalues and Eigenvectors

To find the eigenvalues and eigenvectors of a matrix, follow these steps:

3.1 Step 1: Write the Eigenvalue Equation

The equation for finding eigenvectors and eigenvalues is:

$$A\mathbf{v} = \lambda\mathbf{v}$$

Rearrange this equation as:

$$(A - \lambda I)\mathbf{v} = 0$$

Here, I is the identity matrix. The equation $(A - \lambda I)\mathbf{v} = 0$ is what we call a **homogeneous system**, meaning the left-hand side equals zero. For this system to have non-trivial (non-zero) solutions, the determinant of the matrix $A - \lambda I$ must be zero:

$$\det(A - \lambda I) = 0$$

This is called the **characteristic equation** and is the key to finding eigenvalues.

3.2 Step 2: Solve the Characteristic Equation

Solving $\det(A-\lambda I) = 0$ will give us the eigenvalues. Each eigenvalue represents a different scalar λ, and for each λ, we can compute the corresponding eigenvector.

3.3 Step 3: Find the Eigenvectors

For each eigenvalue, substitute λ into the equation $(A - \lambda I)\mathbf{v} = 0$. This will give a system of linear equations that you can solve to find the eigenvector $\mathbf{v}$.

4 Examples of Finding Eigenvalues and Eigenvectors

Let's take an example where we can compute both the eigenvalues and eigenvectors step-by-step. Consider the matrix:

$$A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$$

4.1 Step 1: Set up the Characteristic Equation

First, we compute $A - \lambda I$:

$$A - \lambda I = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4-\lambda & 1 \\ 2 & 3-\lambda \end{bmatrix}$$

Next, calculate the determinant of $A - \lambda I$:

$$\det(A - \lambda I) = (4-\lambda)(3-\lambda) - (1 \times 2)$$

$$\det(A - \lambda I) = \lambda^2 - 7\lambda + 10 - 2 = \lambda^2 - 7\lambda + 8$$

Setting this equal to zero gives the characteristic equation:

$$\lambda^2 - 7\lambda + 8 = 0$$

Solve for λ using the quadratic formula:

$$\lambda = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(1)(8)}}{2(1)} = \frac{7 \pm \sqrt{49-32}}{2} = \frac{7 \pm \sqrt{17}}{2}$$

Thus, the eigenvalues are:

$$\lambda_1 = \frac{7+\sqrt{17}}{2}, \quad \lambda_2 = \frac{7-\sqrt{17}}{2}$$

4.2 Step 2: Solve for Eigenvectors

For each eigenvalue, we plug λ_1 and λ_2 back into $(A - \lambda I)\mathbf{v} = 0$ and solve for the eigenvectors.

Let's use approximate values for simplicity:

$$\lambda_1 \approx 5.56, \quad \lambda_2 \approx 1.44$$

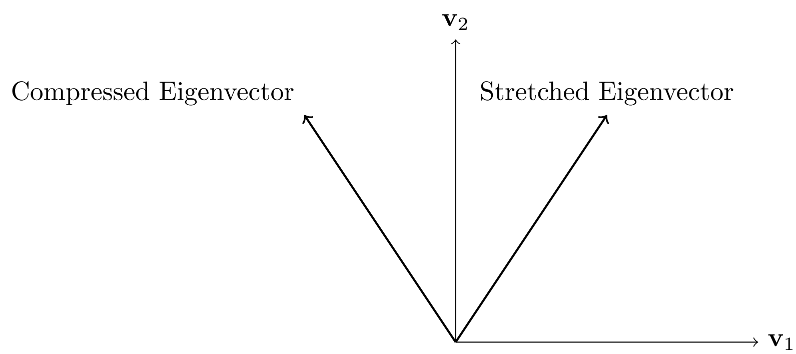
For λ_1, the system becomes:

$$\begin{bmatrix} 4-5.56 & 1 \\ 2 & 3-5.56 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

Simplifying and solving the system gives the corresponding eigenvector.

5 Geometric Interpretation

Eigenvectors represent directions in space. Eigenvalues tell us how much vectors are stretched or compressed in those directions. For example, if we think of the matrix as transforming a grid of points in space, the eigenvectors point along the axes where the transformation only stretches or shrinks points but does not rotate them.



In the diagram, eigenvector $\mathbf{v}_1$ is stretched by the eigenvalue, while eigenvector $\mathbf{v}_2$ is compressed.

6 Applications in Computer Science

Eigenvalues and eigenvectors have several applications:

- **Principal Component Analysis (PCA)**: A technique used in machine learning and statistics to reduce the dimensionality of data. The eigenvectors represent the principal directions of the data, and the eigenvalues represent the variance in those directions.
- **Google PageRank**: The PageRank algorithm ranks websites based on the eigenvector of a matrix that represents the links between pages.
- **Differential Equations**: In physics and engineering, systems of differential equations can often be solved using eigenvalues and eigenvectors.
- **Image Compression**: Eigenvectors can be used to represent important features in image processing, making it possible to compress images while retaining their essential characteristics.

7 Advanced Examples: Complex Eigenvalues

Not all matrices have real eigenvalues. When dealing with complex numbers, the process remains the same, but the eigenvalues and eigenvectors might have imaginary parts.

For example, consider the matrix:

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

The characteristic equation for this matrix is:

$$\det(A - \lambda I) = \lambda^2 + 1 = 0$$

Solving this gives:

$$\lambda = i, \quad \lambda = -i$$

The eigenvalues are complex, and solving for the eigenvectors will also involve complex numbers.

8 Conclusion

Eigenvalues and eigenvectors are powerful tools in mathematics and have important applications in computer science, physics, engineering, and beyond. From simplifying data in machine learning to solving complex systems in physics, understanding eigenvalues and eigenvectors is essential. This document has provided a detailed breakdown of how to compute them, with examples and real-world applications to help illustrate their importance.