CM1015 Set of Combined Transformation Exercises

Recall that when applying two transformations represented by matrices A and B to a vector $\mathbf{v}$, you can either apply them sequentially as

$$\mathbf{v}' = B(A\,\mathbf{v}),$$

or combine them into one matrix $C = BA$ so that

$$\mathbf{v}' = C\,\mathbf{v}.$$

Below are three groups of exercises.

I. Simple Combined Transformations: Rotation and Scaling

Here, the transformation A rotates the vector and B scales it.

1. **Problem 1:** Let

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{(rotation by } 90^\circ\text{)},$$

$$B = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \quad \text{(scaling by 3)},$$

and

$$\mathbf{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

Find $B(A\,\mathbf{v})$ and the combined matrix $C = BA$.

Answer: $A\,\mathbf{v} = \begin{bmatrix} -(-1) \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Then, $B(A\,\mathbf{v}) = \begin{bmatrix} 3 \cdot 1 \\ 3 \cdot 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$. The combined matrix is

$$C = BA = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix},$$

so that $C\,\mathbf{v} = \begin{bmatrix} 0 \cdot 2 + (-3)(-1) \\ 3 \cdot 2 + 0 \cdot (-1) \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$.

2. **Problem 2:** Let

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \quad \text{(scaling by 0.5)},$$

and

$$\mathbf{v} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}.$$

Compute the transformed vector $B(A\,\mathbf{v})$ and find $C = BA$.

Answer: $A\,\mathbf{v} = \begin{bmatrix} -4 \\ -3 \end{bmatrix}$ so that $B(A\,\mathbf{v}) = \begin{bmatrix} 0.5(-4) \\ 0.5(-3) \end{bmatrix} = \begin{bmatrix} -2 \\ -1.5 \end{bmatrix}$. The combined matrix is

$$C = BA = \begin{bmatrix} 0 & -0.5 \\ 0.5 & 0 \end{bmatrix},$$

and indeed, $C\,\mathbf{v} = \begin{bmatrix} 0 \cdot (-3) + (-0.5)(4) \\ 0.5(-3) + 0 \cdot 4 \end{bmatrix} = \begin{bmatrix} -2 \\ -1.5 \end{bmatrix}$.

3. **Problem 3:** Consider a rotation by 180°:

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$

and scaling by 4:

$$B = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}.$$

Let

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Find the result of $B(A\,\mathbf{v})$ and $C = BA$.

Answer: $A\,\mathbf{v} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$ and $B(A\,\mathbf{v}) = \begin{bmatrix} -4 \\ -8 \end{bmatrix}$. The combined matrix is

$$C = BA = \begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix},$$

so that $C\,\mathbf{v} = \begin{bmatrix} -4 \\ -8 \end{bmatrix}$.

4. **Problem 4:** Let

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix},$$

and

$$\mathbf{v} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}.$$

Determine $B(A\,\mathbf{v})$ and the combined transformation $C = BA$.

Answer: $A\,\mathbf{v} = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$; then $B(A\,\mathbf{v}) = \begin{bmatrix} -6 \\ 6 \end{bmatrix}$. The combined matrix is

$$C = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix},$$

so that $C\,\mathbf{v} = \begin{bmatrix} -6 \\ 6 \end{bmatrix}$.

5. **Problem 5:** Let

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix},$$

and

$$\mathbf{v} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

Compute the result of the combined transformation $B(A\,\mathbf{v})$ and find $C = BA$.

Answer: $A\,\mathbf{v} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ so that $B(A\,\mathbf{v}) = \begin{bmatrix} 0 \\ -5 \end{bmatrix}$. The combined matrix is

$$C = BA = \begin{bmatrix} 0 & -5 \\ 5 & 0 \end{bmatrix},$$

yielding $C\,\mathbf{v} = \begin{bmatrix} 0 \\ -5 \end{bmatrix}$.

II. Intermediate Combined Transformations: Reflection and Shear

In these problems, a reflection R is applied first followed by a shear S.

1. **Problem 1:** Let reflection across the y-axis be

$$R = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$$

and let a vertical shear be

$$S = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}.$$

For

$$\mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix},$$

compute $S(R\,\mathbf{v})$ and find the combined matrix $C = SR$.

Answer: $R\,\mathbf{v} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$; then $S(R\,\mathbf{v}) = \begin{bmatrix} -2 \\ 2(-2)+3 \end{bmatrix} = \begin{bmatrix} -2 \\ -4+3 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$. The combined matrix is

$$C = SR = \begin{bmatrix} -1 & 0 \\ -2 & 1 \end{bmatrix}.$$

2. **Problem 2:** Let reflection across the line $y = x$ be given by

$$R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

and let a horizontal shear with factor 3 be

$$S = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}.$$

For

$$\mathbf{v} = \begin{bmatrix} 1 \\ 5 \end{bmatrix},$$

compute $S(R\,\mathbf{v})$ and the combined matrix $C = SR$.

Answer: $R\,\mathbf{v} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$; then $S(R\,\mathbf{v}) = \begin{bmatrix} 5+3\cdot 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \end{bmatrix}$. The combined matrix is

$$C = \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix},$$

noting that when multiplying S and R one obtains the same effect.

3. **Problem 3:** Let reflection across the x-axis be

$$R = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

and let a vertical shear with factor -2 be

$$S = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}.$$

For

$$\mathbf{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix},$$

compute $S(R\,\mathbf{v})$.

Answer: $R\,\mathbf{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$; then $S(R\,\mathbf{v}) = \begin{bmatrix} 3 \\ -2 \cdot 3 + 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 + 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$.

4. **Problem 4:** Let reflection across the line $y = -x$ be

$$R = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix},$$

and let a horizontal shear with factor 1 be

$$S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

For

$$\mathbf{v} = \begin{bmatrix} -2 \\ 4 \end{bmatrix},$$

compute $S(R\,\mathbf{v})$.

Answer: $R\,\mathbf{v} = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$; then $S(R\,\mathbf{v}) = \begin{bmatrix} -4 + 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$.

5. **Problem 5:** Let reflection across the x-axis be

$$R = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

and let a horizontal shear with factor -1 be

$$S = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

For

$$\mathbf{v} = \begin{bmatrix} 4 \\ 0 \end{bmatrix},$$

compute $S(R\,\mathbf{v})$.

Answer: $R\,\mathbf{v} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$ (since the y-coordinate is 0); then $S(R\,\mathbf{v}) = \begin{bmatrix} 4 + (-1)(0) \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$.

III. Advanced Combined Transformations: Rotation and Non-Uniform Scaling

In these exercises, a rotation R is combined with a non-uniform scaling S.

1. **Problem 1:** Let rotation by 60° be given by

$$R = \begin{bmatrix} 0.5 & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & 0.5 \end{bmatrix},$$

and non-uniform scaling

$$S = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}.$$

For

$$\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

compute $S(R\,\mathbf{v})$.
Answer: First,

$$R\,\mathbf{v} = \begin{bmatrix} 0.5 \cdot 2 - \frac{\sqrt{3}}{2} \cdot 1 \\ \frac{\sqrt{3}}{2} \cdot 2 + 0.5 \cdot 1 \end{bmatrix} = \begin{bmatrix} 1 - \frac{\sqrt{3}}{2} \\ \sqrt{3} + 0.5 \end{bmatrix}.$$

Then,

$$S(R\,\mathbf{v}) = \begin{bmatrix} 3\left(1 - \frac{\sqrt{3}}{2}\right) \\ \sqrt{3} + 0.5 \end{bmatrix}.$$

2. **Problem 2:** Let rotation by 120° be given by

$$R = \begin{bmatrix} -0.5 & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -0.5 \end{bmatrix},$$

and non-uniform scaling

$$S = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}.$$

For

$$\mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

compute $S(R\,\mathbf{v})$.
Answer: First,

$$R\,\mathbf{v} = \begin{bmatrix} -0.5 \cdot 1 - \frac{\sqrt{3}}{2}(-1) \\ \frac{\sqrt{3}}{2} \cdot 1 - 0.5(-1) \end{bmatrix} = \begin{bmatrix} -0.5 + \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} + 0.5 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}-1}{2} \\ \frac{\sqrt{3}+1}{2} \end{bmatrix}.$$

Then,

$$S(R\,\mathbf{v}) = \begin{bmatrix} 2 \cdot \frac{\sqrt{3}-1}{2} \\ 4 \cdot \frac{\sqrt{3}+1}{2} \end{bmatrix} = \begin{bmatrix} \sqrt{3} - 1 \\ 2(\sqrt{3}+1) \end{bmatrix}.$$

3. **Problem 3:** Let rotation by 90° be
$$R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$
and non-uniform scaling
$$S = \begin{bmatrix} 0.5 & 0 \\ 0 & 2 \end{bmatrix}.$$
For
$$\mathbf{v} = \begin{bmatrix} 4 \\ 3 \end{bmatrix},$$
compute $S(R\,\mathbf{v})$.
Answer: $R\,\mathbf{v} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$; then
$$S(R\,\mathbf{v}) = \begin{bmatrix} 0.5(-3) \\ 2(4) \end{bmatrix} = \begin{bmatrix} -1.5 \\ 8 \end{bmatrix}.$$

4. **Problem 4:** Let rotation by 45° be
$$R = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix},$$
and non-uniform scaling
$$S = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}.$$
For
$$\mathbf{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix},$$
compute $S(R\,\mathbf{v})$.
Answer: First,
$$R\,\mathbf{v} = \begin{bmatrix} \frac{\sqrt{2}}{2}(-1) - \frac{\sqrt{2}}{2}(2) \\ \frac{\sqrt{2}}{2}(-1) + \frac{\sqrt{2}}{2}(2) \end{bmatrix} = \begin{bmatrix} -\frac{3\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}.$$
Then,
$$S(R\,\mathbf{v}) = \begin{bmatrix} -\frac{3\sqrt{2}}{2} \\ 3 \cdot \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} -\frac{3\sqrt{2}}{2} \\ \frac{3\sqrt{2}}{2} \end{bmatrix}.$$

5. **Problem 5:** Let rotation by 30° be
$$R = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix},$$
and non-uniform scaling
$$S = \begin{bmatrix} 4 & 0 \\ 0 & 0.25 \end{bmatrix}.$$
For
$$\mathbf{v} = \begin{bmatrix} 0 \\ 4 \end{bmatrix},$$

compute $S(R\,\mathbf{v})$.

Answer: $R\,\mathbf{v} = \begin{bmatrix} \frac{\sqrt{3}}{2} \cdot 0 - \frac{1}{2} \cdot 4 \\ \frac{1}{2} \cdot 0 + \frac{\sqrt{3}}{2} \cdot 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 2\sqrt{3} \end{bmatrix}$. Then,

$$S(R\,\mathbf{v}) = \begin{bmatrix} 4(-2) \\ 0.25(2\sqrt{3}) \end{bmatrix} = \begin{bmatrix} -8 \\ \frac{\sqrt{3}}{2} \end{bmatrix}.$$