CM1015 Combined Transformations: Examples and Exercises

Overview

When you have two transformation matrices you can either apply them sequentially or combine them into one matrix. For instance, if you first apply a transformation A and then B (i.e. $B(Av)$ for a vector v), you can combine them into one matrix

$$C = BA,$$

so that

$$Cv = B(Av).$$

In the sections below we consider three cases:

1. **Simple:** Rotation and Scaling.
2. **Intermediate:** Reflection and Shear.
3. **Advanced:** Rotation and Non-Uniform Scaling.

After a brief example explanation for each, you will find five exercises with the answers shown in red.

1 Simple Combined Transformations: Rotation and Scaling

We use:

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{(rotation by } 90^\circ \text{ counterclockwise)}$$

and a scaling matrix

$$B = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix},$$

where k is the scaling factor. The combined transformation is $C = BA$.

Exercises (Simple)

1. **Problem 1:** Given

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Compute $A\mathbf{v}$, then $B(A\mathbf{v})$, and find the combined matrix $C = BA$. **Answer:** $A\mathbf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$; hence $B(A\mathbf{v}) = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$. The combined matrix is $C = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$, so $C\mathbf{v} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$.

2. **Problem 2:** Let

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Find the transformed vector using the combined transformation. **Answer:** $A\mathbf{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $B(A\mathbf{v}) = \begin{bmatrix} -3 \\ 6 \end{bmatrix}$. The combined matrix is $C = \begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix}$, so $C\mathbf{v} = \begin{bmatrix} -3 \\ 6 \end{bmatrix}$.

3. **Problem 3:** Now use a rotation by 180° and scaling by 2. Let

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}.$$

Compute the result. **Answer:** $A\mathbf{v} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ and $B(A\mathbf{v}) = \begin{bmatrix} -6 \\ 2 \end{bmatrix}$. The combined matrix is $C = BA = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$.

4. **Problem 4:** Given

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 4 \\ 2 \end{bmatrix},$$

determine the transformed vector. **Answer:** $A\mathbf{v} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$ and $B(A\mathbf{v}) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. The combined matrix is $C = \begin{bmatrix} 0 & -0.5 \\ 0.5 & 0 \end{bmatrix}$.

5. **Problem 5:** With

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -1 \\ 3 \end{bmatrix},$$

compute the result of the combined transformation. **Answer:** $A\mathbf{v} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$ and $B(A\mathbf{v}) = \begin{bmatrix} -6 \\ -2 \end{bmatrix}$. The combined matrix remains $C = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$, so $C\mathbf{v} = \begin{bmatrix} -6 \\ -2 \end{bmatrix}$.

2 Intermediate Combined Transformations: Reflection and Shear

Here we use a reflection across the x-axis:

$$R = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

and a shear transformation that adds a multiple of the y-coordinate to the x-coordinate:

$$S = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix},$$

so that the combined transformation is $C = SR$.

Exercises (Intermediate)

1. **Problem 1:** Given
$$R = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix},$$
compute the combined transformation. **Answer:** $R\mathbf{v} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$; then $S(R\mathbf{v}) = \begin{bmatrix} 3+2(-1) \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. The combined matrix is $SR = \begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix}$.

2. **Problem 2:** With
$$R = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -2 \\ 4 \end{bmatrix},$$
find the result of the transformation. **Answer:** $R\mathbf{v} = \begin{bmatrix} -2 \\ -4 \end{bmatrix}$; then $S(R\mathbf{v}) = \begin{bmatrix} -2+2(-4) \\ -4 \end{bmatrix} = \begin{bmatrix} -10 \\ -4 \end{bmatrix}$.

3. **Problem 3:** Now let
$$R = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}.$$
Compute the transformed vector. **Answer:** $R\mathbf{v} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$; then $S(R\mathbf{v}) = \begin{bmatrix} 2+3(-5) \\ -5 \end{bmatrix} = \begin{bmatrix} -13 \\ -5 \end{bmatrix}$. The combined matrix is $SR = \begin{bmatrix} 1 & -3 \\ 0 & -1 \end{bmatrix}$.

4. **Problem 4:** Given
$$R = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix},$$
determine the output vector. **Answer:** $R\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$; then $S(R\mathbf{v}) = \begin{bmatrix} 1+(-1)(2) \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

5. **Problem 5:** With
$$R = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -4 \\ 3 \end{bmatrix},$$
compute the combined transformation. **Answer:** $R\mathbf{v} = \begin{bmatrix} -4 \\ -3 \end{bmatrix}$; then $S(R\mathbf{v}) = \begin{bmatrix} -4+0.5(-3) \\ -3 \end{bmatrix} = \begin{bmatrix} -5.5 \\ -3 \end{bmatrix}$. The combined matrix is $SR = \begin{bmatrix} 1 & -0.5 \\ 0 & -1 \end{bmatrix}$.

3 Advanced Combined Transformations: Rotation and Non-Uniform Scaling

In these examples we combine a rotation with non-uniform scaling. Let

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

be a rotation matrix and

$$S = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

be a non-uniform scaling matrix. Their combined transformation is $C = SR$.

Exercises (Advanced)

1. **Problem 1:** Let $\theta = 30^\circ$ so that $\cos 30^\circ = \frac{\sqrt{3}}{2}$ and $\sin 30^\circ = \frac{1}{2}$. Take

$$R = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}, \quad S = \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

   Find $S(R\mathbf{v})$. **Answer:** First,

$$R\mathbf{v} = \begin{bmatrix} \frac{\sqrt{3}}{2} \cdot 1 - \frac{1}{2} \cdot 2 \\ \frac{1}{2} \cdot 1 + \frac{\sqrt{3}}{2} \cdot 2 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} - 1 \\ \frac{1}{2} + \sqrt{3} \end{bmatrix}.$$

   Then,

$$S(R\mathbf{v}) = \begin{bmatrix} 2\left(\frac{\sqrt{3}}{2} - 1\right) \\ 0.5\left(\frac{1}{2} + \sqrt{3}\right) \end{bmatrix} = \begin{bmatrix} \sqrt{3} - 2 \\ \frac{1}{4} + \frac{\sqrt{3}}{2} \end{bmatrix}.$$

2. **Problem 2:** With the same rotation R as in Problem 1 (30°) and now let

$$S = \begin{bmatrix} 0.5 & 0 \\ 0 & 3 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

   Compute the result of $S(R\mathbf{v})$. **Answer:** Compute

$$R\mathbf{v} = \begin{bmatrix} \frac{\sqrt{3}}{2} \cdot 2 - \frac{1}{2} \cdot 1 \\ \frac{1}{2} \cdot 2 + \frac{\sqrt{3}}{2} \cdot 1 \end{bmatrix} = \begin{bmatrix} \sqrt{3} - 0.5 \\ 1 + \frac{\sqrt{3}}{2} \end{bmatrix}.$$

   Then,

$$S(R\mathbf{v}) = \begin{bmatrix} 0.5\left(\sqrt{3} - 0.5\right) \\ 3\left(1 + \frac{\sqrt{3}}{2}\right) \end{bmatrix} = \begin{bmatrix} 0.5\sqrt{3} - 0.25 \\ 3 + \frac{3\sqrt{3}}{2} \end{bmatrix}.$$

3. **Problem 3:** Let $\theta = 45^\circ$ so that $\cos 45^\circ = \sin 45^\circ = \frac{\sqrt{2}}{2}$. Define

$$R = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}, \quad S = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

   Determine $S(R\mathbf{v})$. **Answer:** $R\mathbf{v} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$ so that

$$S(R\mathbf{v}) = \begin{bmatrix} 3 \cdot \frac{\sqrt{2}}{2} \\ 2 \cdot \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} \frac{3\sqrt{2}}{2} \\ \sqrt{2} \end{bmatrix}.$$

4. **Problem 4:** Let $\theta = 60°$ (with $\cos 60° = 0.5,\ \sin 60° = \frac{\sqrt{3}}{2}$) and take

$$R = \begin{bmatrix} 0.5 & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & 0.5 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Compute the transformed vector. **Answer:** $R\mathbf{v} = \begin{bmatrix} -\frac{\sqrt{3}}{2} \\ 0.5 \end{bmatrix}$ and then

$$S(R\mathbf{v}) = \begin{bmatrix} -\frac{\sqrt{3}}{2} \\ 4 \cdot 0.5 \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{3}}{2} \\ 2 \end{bmatrix}.$$

5. **Problem 5:** With $\theta = 45°$ so that

$$R = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}, \quad S = \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 3 \\ -1 \end{bmatrix},$$

determine $S(R\mathbf{v})$. **Answer:** First,

$$R\mathbf{v} = \begin{bmatrix} \frac{\sqrt{2}}{2} \cdot 3 - \frac{\sqrt{2}}{2} \cdot (-1) \\ \frac{\sqrt{2}}{2} \cdot 3 + \frac{\sqrt{2}}{2} \cdot (-1) \end{bmatrix} = \begin{bmatrix} \frac{3\sqrt{2}+\sqrt{2}}{2} \\ \frac{3\sqrt{2}-\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} 2\sqrt{2} \\ \sqrt{2} \end{bmatrix}.$$

Then,

$$S(R\mathbf{v}) = \begin{bmatrix} 2 \cdot (2\sqrt{2}) \\ 0.5 \cdot (\sqrt{2}) \end{bmatrix} = \begin{bmatrix} 4\sqrt{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}.$$