Self-assignment: Set Theory and Relations

2024

Introduction to Set Theory

Set Theory is a fundamental branch of mathematics that deals with the study of sets, which are collections of objects. It provides the basis for various concepts in computer science, including data structures, algorithms, and databases.

0.1 Sets and Set Notation

A **set** is a collection of distinct objects, considered as an object in its own right. Sets are usually denoted by capital letters and their elements are enclosed in curly braces. For example, a set A with elements 1, 2, and 3 is written as:

$$A = \{1, 2, 3\}$$

0.2 Types of Sets

- **Empty Set**: A set with no elements, denoted by $\emptyset$ or $\{\}$.
- **Finite Set**: A set with a countable number of elements. Example: $B = \{a, b, c\}$.
- **Infinite Set**: A set with an uncountable number of elements. Example: $C = \{1, 2, 3, \ldots\}$.
- **Subset**: Set A is a subset of set B (denoted $A \subseteq B$) if all elements of A are also elements of B.
- **Proper Subset**: Set A is a proper subset of set B (denoted $A \subset B$) if $A \subseteq B$ and $A \neq B$.

0.3 Operations on Sets

- **Union** ($\cup$): The union of sets A and B is the set of elements that are in A, B, or both:
$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$
- **Intersection** ($\cap$): The intersection of sets A and B is the set of elements that are in both A and B:
$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

- **Difference** ($-$): The difference of sets A and B is the set of elements that are in A but not in B:
$$A - B = \{x \mid x \in A \text{ and } x \notin B\}$$
- **Complement**: The complement of set A with respect to the universal set U is the set of elements in U that are not in A:
$$A^c = \{x \mid x \in U \text{ and } x \notin A\}$$

1 Relations

A **relation** between two sets A and B is a subset of the Cartesian product $A \times B$. It describes a relationship between elements of A and elements of B.

1.1 Cartesian Product

The **Cartesian product** of sets A and B is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$:

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$$

1.2 Definition of a Relation

A **relation** R from set A to set B is a subset of $A \times B$. If $(a, b) \in R$, we say a is related to b by R, denoted $a\,R\,b$.

1.3 Types of Relations

- **Reflexive**: A relation R on a set A is reflexive if every element is related to itself:
$$\forall a \in A,\ (a, a) \in R$$
- **Symmetric**: A relation R on a set A is symmetric if for every $(a, b) \in R$, $(b, a) \in R$:
$$\forall a, b \in A,\ (a, b) \in R \implies (b, a) \in R$$
- **Transitive**: A relation R on a set A is transitive if for every $(a, b) \in R$ and $(b, c) \in R$, $(a, c) \in R$:
$$\forall a, b, c \in A,\ (a, b) \in R \text{ and } (b, c) \in R \implies (a, c) \in R$$
- **Anti-symmetric**: A relation R on a set A is anti-symmetric if for every $(a, b) \in R$ and $(b, a) \in R$, $a = b$:
$$\forall a, b \in A,\ (a, b) \in R \text{ and } (b, a) \in R \implies a = b$$

1.4 Examples

- Let $A = \{1, 2, 3\}$ and $B = \{a, b\}$. The Cartesian product $A \times B$ is:

$$A \times B = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$$

- Consider the relation R on A where $R = \{(1, 1), (2, 2), (3, 3)\}$. This relation is reflexive, symmetric, and transitive.
- For a relation R on $A = \{1, 2\}$ defined as $R = \{(1, 2), (2, 1)\}$, the relation is symmetric but not reflexive or transitive.

Conclusion

Set theory and relations are foundational concepts in computer science. Understanding sets, their operations, and relations between elements provides a basis for more advanced topics such as databases, algorithms, and formal languages. Mastery of these topics is crucial for analyzing and designing complex systems.

References

- MIT OpenCourseWare: Real Analysis
- MIT OpenCourseWare: Mathematics for Computer Science