Self-assignment: Proof Techniques

2024

Introduction to Proof Techniques

Proof techniques are fundamental methods used in mathematics and computer science to establish the validity of statements. The primary techniques include direct proof, proof by contradiction, and proof by induction. Each method has its own set of rules and applications, making them essential tools for rigorous reasoning.

1 Direct Proof

A **direct proof** involves showing that a statement is true by a straightforward chain of logical deductions from known facts or axioms. The goal is to directly establish the truth of the statement through logical reasoning.

1.1 Structure of a Direct Proof

1. State the theorem or proposition: Clearly define what needs to be proved.
2. Assume the premises: Start from the given conditions or hypotheses.
3. Logical deductions: Use definitions, theorems, and logical operations to derive the conclusion.
4. Conclude: Demonstrate that the conclusion follows logically from the premises.

1.2 Example

Theorem: If n is an even integer, then n^2 is also even.

Proof: Let n be an even integer. By definition, $n = 2k$ for some integer k. Then:

$$n^2 = (2k)^2 = 4k^2 = 2(2k^2)$$

Since $2k^2$ is an integer, n^2 is of the form $2 \times \text{(integer)}$, which means n^2 is even. Therefore, the theorem is proved.

2 Proof by Contradiction

A **proof by contradiction** (also known as an indirect proof) involves assuming the negation of the statement to be proved and showing that this assumption leads to a contradiction. If a contradiction is reached, the original statement is proven to be true.

2.1 Structure of a Proof by Contradiction

1. State the theorem: Define the statement to be proved.
2. Assume the negation: Assume that the statement is false.
3. Derive a contradiction: Use logical reasoning to derive a contradiction from the assumption.
4. Conclude: Since the assumption leads to a contradiction, the original statement must be true.

2.2 Example

Theorem: There is no largest prime number.

Proof: Assume, for contradiction, that there is a largest prime number p. Consider the number $N = p! + 1$ (where $p!$ denotes the factorial of p).

By construction, N is not divisible by any prime number less than or equal to p because when divided by any such prime, N leaves a remainder of 1. Thus, N must be either prime itself or divisible by some prime larger than p. In either case, this contradicts the assumption that p is the largest prime number. Therefore, our assumption is false, and there is no largest prime number.

3 Proof by Induction

A **proof by induction** is a method used to prove statements about integers. It involves proving a base case and then proving that if the statement holds for an arbitrary integer k, it also holds for $k + 1$.

3.1 Structure of a Proof by Induction

1. Base Case: Verify that the statement holds for the initial value, usually $n = 0$ or $n = 1$.
2. Inductive Step: Assume the statement holds for an arbitrary integer k (inductive hypothesis). Show that if the statement holds for k, it also holds for $k + 1$.
3. Conclusion: By the principle of mathematical induction, the statement is true for all integers greater than or equal to the base case.

3.2 Example

Theorem: For all integers $n \geq 1$, the sum of the first n positive integers is given by:

$$S(n) = \frac{n(n+1)}{2}$$

Proof: **Base Case:** For $n = 1$, the sum $S(1) = 1$ and $\frac{1(1+1)}{2} = 1$. The base case holds.

Inductive Step: Assume the formula holds for $n = k$, i.e.,

$$S(k) = \frac{k(k+1)}{2}$$

We need to show it holds for $n = k + 1$. The sum of the first $k + 1$ integers is:

$$S(k+1) = S(k) + (k+1)$$

Using the inductive hypothesis:

$$S(k+1) = \frac{k(k+1)}{2} + (k+1)$$

Combine terms:

$$S(k+1) = \frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$$

This matches the formula for $n = k + 1$. Hence, by induction, the formula is true for all $n \geq 1$.

Conclusion

Proof techniques are essential tools in mathematics and computer science. Direct proofs offer straightforward logical reasoning, proofs by contradiction demonstrate the truth by negating assumptions, and proofs by induction establish statements for all integers through a base case and inductive reasoning. Mastery of these techniques is crucial for rigorous mathematical and algorithmic reasoning.

References

- MathWorld: Proof by Induction
- ProofWiki: Proof by Contradiction
- Math StackExchange: Direct Proofs