UNIVERSITY OF LONDON

GOLDSMITHS COLLEGE

Department of Computing

B. Sc. Examination 2018

IS51002E
Mathematical Modelling for Problem Solving

Duration: 3 hours

Date and time:

This paper is in two parts: part A and part B. You should answer ALL questions from part A and THREE questions from part B. Part A carries 40 marks, and each question from part B carries 20 marks. The marks for each part of a question are indicated at the end of the part in [.] brackets.

There are 100 marks available on this paper.

Electronic calculators must not be programmed prior to the examination. Calculators which display graphics, text or algebraic equations are not allowed.

THIS PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM

Part A

Multiple choice

Question 1 Each question has one or more correct answers

(a) Let $A = \{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}\}$. Which of the following sets represent A using the inclusion rules? More than one answer may apply.

i. $\{2^{-n} : n \in \mathcal{Z}$ **and** $0 \leq n \leq 7\}$

ii. $\{2^{-n} : n \in \mathcal{Z}$ **and** $0 \leq n < 8\}$

iii. $\{\frac{1}{2n} : n \in \mathcal{Z}$ **and** $0 \leq n \leq 7\}$

iv. $\{\frac{1}{2n} : n \in \mathcal{Z}$ **and** $0 < n < 8\}$

[2]

(b) Let $S = \{1, 2, 3\}$, which one of the following sets represents $\mathcal{P}(S)$?

i. $\{\{1\}, \{2\}, \{3\}\}$

ii. $\{\{1\}, \{2\}, \{3\}, \{1, 2\}, 1, 3, \{2, 3\}\}$

iii. $\{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$

iv. $\{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$

[2]

(c) Let p and q and be two propositions where p means '**Jack is happy**' and q means '**Jack paints a picture**'. Which one of the following logical expressions is a correct formalisation of the following sentence:

Jack is happy only if he paints a picture.

i. $p \rightarrow q$

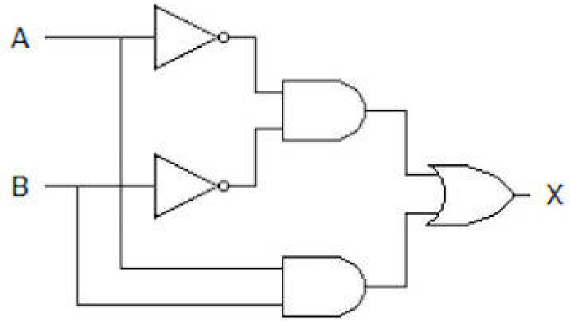
ii. $q \rightarrow p$

iii. $p \wedge q$

iv. $p \rightarrow \neg q$

[2]

(d) Which one is a correct output of the following logic network:



i. $(A \wedge B) \vee (\neg A \wedge \neg B)$

ii. $(A \wedge B) \vee (\neg A \wedge B)$

iii. $(A \wedge B) \vee (A \wedge \neg B)$

iv. $(A \vee B) \wedge (\neg A \vee \neg B)$

[2]

(e) Let $f : R^+ \rightarrow R$ be a function where $f(x) = \log_2 x$. Which one of the following is the inverse function of the function f?

i. $f^{-1}(x) = 2^x$

ii. $f^{-1}(x) = e^x$

iii. $f^{-1}(x) = \sqrt{x}$

iv. $f^{-1}(x) = \frac{x}{2}$

[2]

(f) The following sequence $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \cdots$ is

i. arithmetic

ii. geometric

iii. neither geometric nor arithmetic

iv. both arithmetic and geometric

[2]

(g) Let p and q be two propositions. Which one of the following compound statements is equivalent to $\neg(p \wedge q)$?

i. $\neg p \wedge \neg q$
ii. $\neg p \vee \neg q$
iii. $p \wedge q$
iv. $p \oplus q$

[2]

(h) Which one of the following correctly describes a complete graph G?

i. G is a simple graph where every two vertices has a direct link between them
ii. G is a simple graph connected graph
iii. G is a graph with parallel edges between every two vertices.
iv. none of the above

[2]

(i) Which of the following statements is/are **TRUE**? More than one answer might apply.

i. it is possible to draw a 3-regular graph with 5 vertices
ii. it is possible to draw 3-regular graph with 6 vertices
iii. the sum of the degree sequence of a graph is twice the number of edges in the graph
iv. the sum of the degree sequence of a graph is twice the number of vertices in the graph.

[2]

(j) The degree of each vertex in complete graph k_n is

i. n-2
ii. n-1
iii. n
iv. 2n

[2]

(k) What is the decimal representation of 321_8?

i. 83_{10}
ii. 418_{10}
iii. 209_{10}
iv. none of the above

[2]

(l) What is the multiplicative inverse of 5 in modulo 7?

i. 1
ii. 2
iii. 3
iv. 4

[2]

(m) A triangle XYZ has sides $x = 8$, $y = 7$ and angle $Y = 1.13$ radians. The size of angle X is:

i. 0.441
ii. 1.111
iii. 0.913
iv. This triangle does not exist

[2]

(n) Convert 1.7 radians to degrees

i. 97.4^o
ii. 48.7^o
iii. 194.8^o
iv. 33.7^o

[2]

(o) The frequency of $f(x) = 2\cos(\pi + x)$ is

i. π
ii. 4π
iii. 2π
iv. $\frac{1}{2\pi}$

[2]

(p) $\log_2 6 + \log_2 \frac{1}{2}$ is equal to:

i. 6.5
ii. $\log_2 6.5$
iii. $\log_2 3$
iv. 3

[2]

(q) The graph of $\log_2 x$:

i. has a x-intercept of 1
ii. has a y-intercept of 0
iii. passes through the point $(1, 2)$
iv. passes through the point $(0, 0)$

[2]

(r) Calculate the following limit: $\lim_{x\to\infty} \frac{x^5+x^3-7}{2x^5-3x+1}$.

i. -7
ii. ∞
iii. $\frac{1}{2}$
iv. is not defined

[2]

(s) Given $y = x^2(x^2 + x)$

i. $\frac{dy}{dx}$=$x^4 + x^3$
ii. $\frac{dy}{dx}$=$2x(2x + 1)$

iii. $\frac{dy}{dx}$=$4x^3 + 3x^2$

iv. $\frac{dy}{dx}$ is not defined

[2]

(t) Convert the vector $\vec{u}$=$\begin{pmatrix} 2 \\ 5 \end{pmatrix}$ in cartesian coordinates to polar coordinates

i. $(4.58, 1.19)$

ii. $(5.39, 1.19)$

iii. $\sqrt{21}$

iv. $\sqrt{29}$

[2]

Part B

Question 2 Set, Logic & Sequences

(a) i. Describe the set A by the listing method.

$$A = \{r^3 - 1 : r \in \mathcal{Z} \textbf{ and } -1 < r \leq 3\}.$$

ii. Describe the set B by the rule of inclusion method where $B = \{1, 2, 4, 8, 16, \cdots, 64\}$.

[2]

(b) Let A and B and C be subsets of a universal set $\mathcal{U}$.

1. Draw a labelled Venn diagram depicting A, B, C in such a way that they divide $\mathcal{U}$ into 8 disjoint regions. [1]
2. The subset $X \subseteq \mathcal{U}$ is defined by the following membership table:

A	B	C	X
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	0

Shade the region X on your diagram. Describe the region you have shaded in set notation as simply as you can. [3]

(c) Let p and q be the following propositions concerning a positive integer n:

p : *'n has one digit'*

q : *'n is less than 10'.*

i. Express each of the three following compound propositions concerning positive integers symbolically by using p, q and appropriate logical symbols.

'n has one digit if n is less than 10'

'n has one digit only if n is less than 10'

'n has one digit or greater than or equal to 10 but not both'

[3]

ii. Construct the truth table for the statement $q \to p$. [2]

iii. Write in words the contrapositive of the statement given symbolically by
'$q \to p$'. [2]

(d) i. Express the following sum using the $\sum$ notation [1]

$$1 + 3 + 5 + 7 + ... + (2n - 1)$$

ii. Evaluate the following the following sum: [2]

$$\sum_{k=21}^{100} 4k$$

Hint: you might want to use the formula: $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

iii. Let $S_n = 1 + 2 + 3 + \cdots + n, \;\; for\ n \geq 1$.

1. Calculate S_1, S_2. [1]
2. Prove by induction that: $S_n = \frac{n(n+1)}{2}, \quad \forall\, n \geq 1$. [3]

Question 3 Graphs, Trees & Relations

(a) i. Is it possible to construct a 3-regular graph with 7 vertices ? Explain your answer. [1]

ii. Is it possible to construct a simple graph with the degree sequence 4,3,2,2? Explain your answer. [1]

iii. A graph, G, with 5 vertices: $a,\ b,\ c,\ d,\ e$ has the following adjacency list:

$a : b,\ e$

$b : a,\ c,\ d$

$c : b,\ d$

$d : b,\ c,\ e$

$e : d,\ a.$

1. Draw the graph, G. [2]
2. Write down the degree sequence of G. State the relationship between the number of edges in G and its corresponding degree sequence. [1]

   Draw two non-isomorphic spanning trees of G. [2]

(b) i. Define what a tree is. [1]

ii. How many edges in a trees with n vertices? [1]

iii. A binary search tree is designed to store an ordered list of 4000 records, numbered 1,2,3,...,4000 at its internal nodes. Draw levels 0, 1 and 2 of this tree, showing which number record is stored at the root and at each of the nodes at level 1 and 2, making it clear which records are at each level and find the height of this tree? [5]

(c) Given S be the set of integers $\{1, 2, 3, 4, 5, 6\}$. Let $\mathcal{R}$ be a relation defined on S by the following condition such that, for all $x, y \in S$, xRy if $x\ mod\ 3 = y\ mod\ 3$.

i. Draw the digraph of $\mathcal{R}$. [2]

ii. Show that $\mathcal{R}$ is an equivalence relation. [3]

iii. Write down the equivalence classes of $\mathcal{R}$. [1]

Question 4 Functions & Graph Sketching

(a) Let $f : \mathcal{R} \to \mathcal{R}$ with $f(x) = x^2 + 1$

i. List the co-domain and the range of f.

ii. Find the ancestors if any of 5.

iii. Is f a one to one function? Explain your answer.

iv. Is f an onto function? Explain your answer.

[5]

(b) Find the following limits:

i. $\lim_{x \to 2} \frac{x^2-1}{x^3-x}$

ii. $\lim_{x \to 0^-} \frac{x^2-1}{x^3-x}$

iii. $\lim_{x \to 0^+} \frac{x^2-1}{x^3-x}$

iv. $\lim_{x \to \infty} \frac{x^2-1}{x^3-x}$ [4]

(c) Given the function $f(x) = (x-1)(x^2+x+1)$

i. Find the value or values of x for which $f(x) = 0$
(note $(x^2+x+1) \geq 0$ for all x)

ii. Differentiate $f(x)$.

iii. Hence find any stationary points of $f(x)$ and determine their nature.

iv. Sketch $f(x)$. [6]

(d) i. Find numerical values for the following

$\log_{10} 0.001$

$\log_{1000} 10$

ii. Give the function $f(x) = 1 + log_2 x$

Plot the graph of $f(x)$

Find the inverse function $f^{-1}(x)$ [5]

Question 5 Bases & Modular Arithmetic

(a) i. Express the decimal number $(177)_{10}$ in base 8. [1]

ii. Express the decimal number $(11.125)_{10}$ as a binary number. [1]

iii. Express the hexadecimal number $(32.8)_{16}$ as a decimal number. [1]

iv. Express the octal number $(262.24)_8$ as

(1) a binary number

(2) a hexadecimal number [2]

v. Working in base 8 and showing all your working, compute the following:

$$(4763)_8 + (332)_8 - (4606)_8$$

[3]

(b) i. Find the smallest positive integer modulo 17 that is congruent to

(1) 271

(2)1277 [2]

ii. Find the remainder on division by 17 of

(1) $271 - 1277$

(2) 271×1277 [2]

iii. Find the following

(1) the additive inverse of 15 modulo 17

(2) the multiplicative inverse of 15 modulo 17 [2]

(c) i. Define what is meant by a rational number. Say whether or not the repeating decimal number is 0.131313... is rational, justify your answer. [2]

ii. Give an example of an irrational number. [1]

iii. Showing all your working, express the recurring decimal 0.272727... as a fraction in its lowest form. [3]

Question 6 Probability, Vectors & Matrices

(a) Two Friends, Jack and Charles, frequently play golf and tennis with each other. In a long run, it has been found that Jack wins 3 rounds of golf out of every 5, and 1 game of tennis out of every 4 games. If they play one round of golf and one game of tennis find the probability that Jack

i. wins both,

ii. loses both,

iii. wins the round of golf only.

iv. wins either the golf round or the tennis game but not both.

[6]

(b) Given $\vec{v}_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$

i. Find the magnitudes of $\vec{v}_1$ and $\vec{v}_2$.

ii. Find the dot product of $\vec{v}_1$ and $\vec{v}_2$.

iii. Hence find the angle between $\vec{v}_1$ and $\vec{v}_2$.

iv. Find $\vec{v}_3$ the cross product (vector product) of $\vec{v}_1$ and $\vec{v}_2$.

[6]

(c) Let A be a 3x3 homogeneous transformation matrix corresponding to a scaling of the x and y-coordinates by a factor of 2 and a factor of 3 respectively, let B be a 3x3 homogeneous transformation matrix corresponding to a translation of the x and y coordinates by 1 and -1 respectively and let C be a 3x3 homogeneous transformation matrix corresponding to a clockwise rotation about the z-axis through an angle $\frac{\pi}{6}$.

i. Find matrices A, B and C. [3]

ii. How would the transformation represented by the matrix B transform the following three points which represent a triangle in the Cartesian space: (1,0), (2,0) and (2,1)? [2]

iii. Find the inverse matrices A^{-1} and C^{-1}. [3]

END OF EXAMINATION