Self-assignment: Combinatorics

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Introduction to Combinatorics

Combinatorics is the branch of mathematics that studies the counting, arrangement, and combination of objects. It plays a vital role in areas such as probability theory, algebra, and computer science. The three main principles are:

- **Counting**: Counting how many ways certain tasks can be performed.
- **Permutations**: Arranging objects in a specific order.
- **Combinations**: Selecting objects from a set without regard to order.

In this guide, we will explore each concept step by step, with examples and explanations.

1 Basic Counting Principles

1.1 Addition Principle

If there are n_1 ways of performing task 1 and n_2 ways of performing task 2, then the number of ways to perform either task 1 or task 2 is:

$$n_1 + n_2$$

Example 1: If you can choose from 3 different shirts and 4 different pants, how many different outfits can you make?

$$\text{Answer: } 3 + 4 = 7$$

1.2 Multiplication Principle

If there are n_1 ways to perform task 1, and for each way of doing task 1, there are n_2 ways of performing task 2, then the total number of ways to perform both tasks is:

$$n_1 \times n_2$$

Example 2: If you can choose from 3 different shirts and 4 different pants, how many different outfits can you make if you choose both a shirt and pants?

$$\text{Answer: } 3 \times 4 = 12$$

2 Permutations

Permutations refer to the arrangement of objects where the order matters. The number of ways to arrange n distinct objects is given by:

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 1$$

Example: How many ways can you arrange the letters in the word "COMB"?

$$\text{Answer: } 4! = 4 \times 3 \times 2 \times 1 = 24$$

2.1 Permutations with Repetition

If some objects are repeated, the formula for permutations becomes:

$$\frac{n!}{n_1! n_2! \cdots n_k!}$$

where $n_1, n_2, \ldots, n_k$ are the frequencies of the repeated objects.

Example: How many ways can you arrange the letters in the word "LEVEL"?

$$\text{Answer: } \frac{5!}{2!2!1!} = \frac{120}{4} = 30$$

3 Combinations

Combinations refer to selecting objects from a set where the order does not matter. The number of ways to choose r objects from a set of n objects is given by:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Example: How many ways can you choose 3 students from a class of 5?

$$\text{Answer: } \binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{120}{6 \times 2} = 10$$

4 Worked Example

Problem: A committee of 3 people is to be chosen from a group of 6 men and 4 women. In how many ways can the committee be formed if:

- All committee members are men?

- The committee must consist of 2 men and 1 woman?

Solution:

1. To choose 3 men from 6:

$$\binom{6}{3} = \frac{6!}{3!(6-3)!} = \frac{720}{6 \times 6} = 20$$

2. To choose 2 men from 6 and 1 woman from 4:

$$\binom{6}{2} \times \binom{4}{1} = \frac{6!}{2!(6-2)!} \times \frac{4!}{1!(4-1)!} = \frac{720}{2 \times 24} \times 4 = 15 \times 4 = 60$$

Thus, the number of ways to form the committee in each case is:

- **Case 1**: 20 ways.
- **Case 2**: 60 ways.

Conclusion

Combinatorics provides powerful tools for counting and arranging objects. By mastering these fundamental techniques, you will develop the ability to solve complex problems in various fields, including computer science and probability theory.

References

- MIT OpenCourseWare: Combinatorics