MATH

I really can't remember these things

github/jovyntls

01. DIFFERENTIATION

Taylor's Theorem

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \dots$$
$$\dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n,$$

where $R_n = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{(n+1)}$ for c between x and a

Taylor Series

As $R_n \to 0$ as $n \to \infty$, then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Differentiation Techniques

$f(x)$	$f'(x)$
$\tan x$	$\sec^2 x$
$\csc x$	$-\csc x \cot x$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\csc^2 x$
$a^{f(x)}$	$\ln a \cdot f'(x) a^{f(x)}$
$\log_a f(x)$	$\frac{f'(x)}{\ln a \cdot f(x)}$
$\sin^{-1} f(x)$	$\frac{f'(x)}{\sqrt{1-[f(x)]^2}}, \quad \lvert f(x)\rvert<1$
$\cos^{-1} f(x)$	$-\frac{f'(x)}{\sqrt{1-[f(x)]^2}}, \quad \lvert f(x)\rvert<1$
$\tan^{-1} f(x)$	$\frac{f'(x)}{1+[f(x)]^2}$
$\cot^{-1} f(x)$	$-\frac{f'(x)}{1+[f(x)]^2}$
$\sec^{-1} f(x)$	$\frac{f'(x)}{\lvert f(x)\rvert\sqrt{[f(x)]^2-1}}$
$\csc^{-1} f(x)$	$-\frac{f'(x)}{\lvert f(x)\rvert\sqrt{[f(x)]^2-1}}$

derivatives of trigonometric functions

function	derivative	function	derivative
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$	$\csc^{-1} x$	$\frac{-1}{x\sqrt{x^2-1}}$
$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}$	$\sec^{-1} x$	$\frac{1}{x\sqrt{x^2-1}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$	$\cot^{-1} x$	$\frac{-1}{1+x^2}$

logarithmic differentiation I

aka take $\ln$ on both sides and implicitly differentiate

for $y = f_1(x) f_2(x) \cdots f_n(x)$ (product of nonzero functions),

$$\ln|y| = \ln|f_1(x)| + \ln|f_2(x)| + \dots + \ln|f_n(x)|$$
$$\frac{dy}{dx} = \left[\frac{f_1'(x)}{f_1(x)} + \frac{f_2'(x)}{f_2(x)} + \dots + \frac{f_n'(x)}{f_n(x)}\right] y$$
$$= \left[\frac{f_1'(x)}{f_1(x)} + \frac{f_2'(x)}{f_2(x)} + \dots + \frac{f_n'(x)}{f_n(x)}\right] f_1(x) f_2(x) \cdots f_n(x)$$

logarithmic differentiation II

for $y = f(x)^{g(x)} (f(x) > 0)$,

$$\ln y = g(x) \ln f(x) \Rightarrow \frac{dy}{dx} = y \frac{d}{dx}[g(x) \ln f(x)]$$

$$\lim_{x\to a} (f(x)^{g(x)}) = \lim_{x \to a} \exp\left(g(x) \ln f(x)\right)$$
$$= \exp\left(\lim_{x\to a} g(x) \ln f(x)\right)$$

02. INTEGRATION

Integration Techniques

$f(x)$	$\int f(x)$
$\tan x$	$\ln(\sec x)$, $\lvert x\rvert < \frac{\pi}{2}$
$\cot x$	$\ln(\sin x)$, $0 < x < \pi$
$\csc x$	$-\ln(\csc x + \cot x)$, $0 < x < \pi$
$\sec x$	$\ln(\sec x + \tan x)$, $\lvert x\rvert < \frac{\pi}{2}$
$\frac{1}{x^2+a^2}$	$\frac{1}{a}\tan^{-1}(\frac{x}{a})$
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}(\frac{x}{a})$, $\lvert x\rvert < a$
$\frac{1}{x^2-a^2}$	$\frac{1}{2a}\ln(\frac{x-a}{x+a})$, $x > a$
$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\ln(\frac{x+a}{x-a})$, $x < a$
a^x	$\frac{a^x}{\ln a}$

Rational functions

for $f = \frac{A(x)}{B(x)}$,

- manipulate such that $\deg A(x) < \deg B(x)$, then decompose into partial fractions
- common rational functions:
  - $$\int \frac{1}{(x+a)^k}\,dx = \begin{cases} \ln|x+a| + K, & \text{if } k = 1 \\ \frac{(x+a)^{1-k}}{1-k} + K, & \text{if } k \geq 1 \end{cases}$$
  - $$\int \frac{u}{(u^2+d^2)^r}\,du = \begin{cases} \frac{1}{2}\ln(u^2+d^2), & \text{if } r = 1 \\ \frac{(u^2+d^2)^{1-r}}{2(1-r)}, & \text{if } r \geq 2 \end{cases}$$
  - $$\int \frac{1}{(u^2+d^2)^r}\,du = \frac{1}{d^{2r-1}} \int \frac{1}{(t^2+1)^r}\,dt$$
  -

partial fractions

- for each linear factor $(x+a)^k$:
  - $\frac{A_1}{x+a} + \frac{A_2}{(x+a)^2} + \dots + \frac{A_k}{(x+a)^k}$
- for each quadratic factor $(x^2+bx+c)^r$:
  - $\frac{B_1x+C_1}{x^2+bx+c} + \dots + \frac{B_rx+C_r}{(x^2+bx+c)^r}$

trigonometric substitutions

- $\sqrt{a^2-x^2}$, $\quad x = a\sin t$, $\quad t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$
- $\sqrt{x^2-a^2}$, $\quad x = a\sec t$, $\quad t \in [0, -\frac{\pi}{2}) \cup (\pi, \frac{3\pi}{2}]$
- a^2+x^2, $\quad x = a\tan t$, $\quad t \in (-\frac{\pi}{2}, \frac{\pi}{2})$

universal trigonometric substitution

any rational expression in $\sin x$ and $\cos x$ can be integrated using the substitution $t = \tan\frac{x}{2}$, $\quad x \in (-\pi, \pi)$.

$$\sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}, \quad \frac{dx}{dt} = \frac{2}{1+t^2}$$

trigonometric identities

- $\tan^{-1} x + \cot^{-1} x - \frac{\pi}{2}$
- $\sec^{-1} x + \csc^{-1} x = \begin{cases} \frac{\pi}{2}, & \text{if } x \geq 1 \\ \frac{5\pi}{2}, & \text{if } x \leq -1 \end{cases}$

03. SERIES

MacLaurin Series

For $-\infty < x < \infty$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$
$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

For $-1 < x < 1$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$
$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$
$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$
$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$$
$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$$
$$\frac{1}{(1+x)^2} = \sum_{n=1}^{\infty} (-1)^{n-1} n x^{n-1}$$
$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}$$
$$\frac{1}{(1-x)^3} = \frac{1}{2}\sum_{n=2}^{\infty} n(n-1) x^{n-2}$$
$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$$
$$= 1 + kx + \frac{k(k-1)}{2!}x^2 + \dots$$

MISC

Exponentials

properties

- $a^u a^v = a^{u+v}$
- $a^{-u} = \frac{1}{a^u}$
- $(a^u)^v = a^{uv}$
- $(a^x)' = a^x \ln a$
- $\frac{d}{dx} x^r = r x^{r-1}$

- $\lim_{x\to\infty} e^x = \infty$, $\lim_{x\to-\infty} e^x = 0$
- $\lim_{x\to\infty} \frac{e^x}{x^n} = \infty$ for $n \in \mathbb{Z}^+$
- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \dots$

- $\int x^r\,dx = \begin{cases} \frac{x^{r+1}}{r+1} + C & \text{if } r \neq -1, \\ \ln x + C & \text{if } r = -1, \end{cases}$
  - if r is irrational, then x^r is only defined for $x \geq 0$.

triangle inequality

$|a+b| \leq |a| + |b|$ for all $a, b \in \mathbb{R}$

binomial theorem

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$
$$= a^n + \binom{n}{1} a^{n-1} b + \dots + \binom{n}{n-1} a b^{n-1} + b^n$$

where the binomial coefficient is given by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

factorisation

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \dots + ab^{n-2} + b^{n-1})$$
$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$
$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

misc

- $\forall x \in (0, \frac{\pi}{2}), \sin x < x < \tan x$
- $\sin\theta = \frac{\tan\theta}{\sqrt{\tan^2\theta + 1}}$