Self Assignment: Vector Calculus

2024

1 Introduction

In this document , we will cover Vector calculus and other topics related to it. Vector Calculus plays a key role in fields such as computer science, physics, and engineering, especially when dealing with 3D space and multidimensional functions. We will cover three fundamental operations: Gradient, Divergence, and Curl. Each of these is essential for understanding the behavior of vector fields.

2 Gradient

The **gradient** of a scalar field $f(x, y, z)$ is a vector field that points in the direction of the greatest rate of increase of the function and whose magnitude is the rate of that increase. It is denoted by ∇f or grad f.

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

2.1 Example

Consider the scalar field $f(x, y, z) = x^2 + y^2 + z^2$. The gradient is computed as:

$$\nabla f = (2x, 2y, 2z)$$

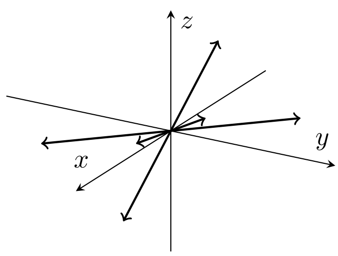
The gradient at the point $(1, 1, 1)$ would be:

$$\nabla f = (2, 2, 2)$$

This shows that at the point $(1, 1, 1)$, the function increases most rapidly in the direction of the vector $(2, 2, 2)$.

2.2 Visualization

The gradient can be visualized as arrows pointing outwards from points on the surface of a 3D function. Below is an illustration of the gradient field for $f(x, y, z) = x^2 + y^2 + z^2$.



3 Divergence

The **divergence** of a vector field $\mathbf{F} = (F_x, F_y, F_z)$ measures the rate at which "stuff" expands out of a point. In mathematical terms, it is the scalar field given by:

$$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

3.1 Example

Let $\mathbf{F}(x, y, z) = (x^2, y^2, z^2)$. Then the divergence is:

$$\nabla \cdot \mathbf{F} = 2x + 2y + 2z$$

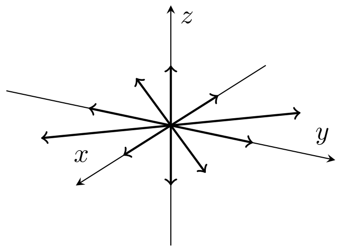
At the point $(1, 1, 1)$, the divergence is:

$$\nabla \cdot \mathbf{F} = 2(1) + 2(1) + 2(1) = 6$$

This means the vector field is "expanding" outwards at this point with a rate of 6.

3.2 Visualization

Divergence can be visualized as the rate at which vectors spread out from a point. A positive divergence indicates outward flow, while negative divergence indicates inward flow. Below is an illustration of a vector field with positive divergence.



4 Curl

The **curl** of a vector field $\mathbf{F} = (F_x, F_y, F_z)$ represents the tendency of the field to rotate around a point. It is denoted by $\nabla \times \mathbf{F}$ and is given by:

$$\nabla \times \mathbf{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right)$$

4.1 Example

Let $\mathbf{F}(x, y, z) = (-y, x, 0)$. Then the curl is:

$$\nabla \times \mathbf{F} = (0 - 0, 0 - 0, 1 - (-1)) = (0, 0, 2)$$

At any point in the plane, the curl is $(0, 0, 2)$, which indicates that the vector field has a constant rotation around the z-axis.

4.2 Visualization

Curl can be visualized as the rotation of vectors around a point, similar to how fluid might circulate in a whirlpool. Below is an illustration of a vector field with a non-zero curl.