Problems for Calculus

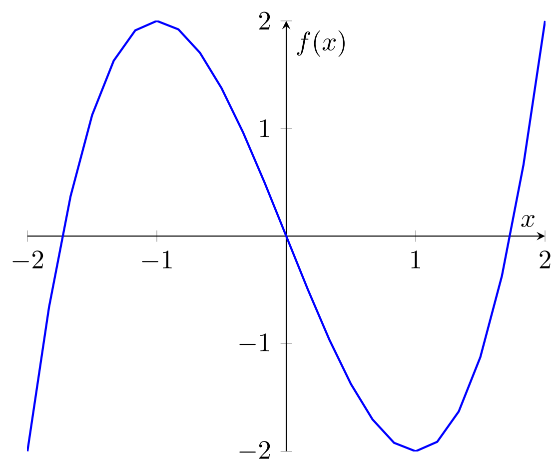
October 28, 2024

Easy Problems

1. Sketch the graph of $f(x) = x^3 - 3x$. Use the first and second derivatives to determine the intervals where the function is increasing, decreasing, concave up, and concave down.

$$f'(x) = 3x^2 - 3,\ f''(x) = 6x$$

Increasing on $x > 1$, decreasing on $x < -1$, concave up for $x > 0$, concave down for $x < 0$.



2. Find the derivative of the piecewise function:

$$f(x) = \begin{cases} x^2 & \text{if } x < 0 \\ 2x + 1 & \text{if } x \geq 0 \end{cases}$$

$$f'(x) = \begin{cases} 2x & \text{if } x < 0 \\ 2 & \text{if } x > 0 \end{cases}$$

3. Compute $\lim_{x\to\infty} \frac{4x^2+1}{2x^2+3x+5}$.

$$\lim_{x\to\infty} \frac{4x^2+1}{2x^2+3x+5} = 2$$

4. Find the critical points of $f(x) = \frac{x^3}{3} - x$ and determine if they are minima or maxima.

$$f'(x) = x^2 - 1 \text{ gives critical points at } x = 1 \text{ and } x = -1.$$

5. Solve $\int (2x^3 - 4x)\,dx$.

$$\int (2x^3 - 4x)\,dx = \frac{x^4}{2} - 2x^2 + C$$

Medium Problems

1. Differentiate $f(x) = x^2 \ln(x)$ using the product rule.

$$f'(x) = 2x\ln(x) + x$$

2. Use the chain rule to differentiate $f(x) = e^{x^2-1}$.

$$f'(x) = 2xe^{x^2-1}$$

3. Solve $\lim_{x\to 0} \frac{\tan(x)}{x}$ using the small-angle approximation.

$$\lim_{x\to 0} \frac{\tan(x)}{x} = 1$$

4. Solve the integral $\int_0^2 (x^3 - 2x)\,dx$.

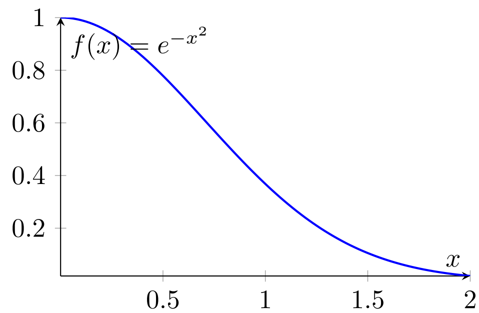
$$\int_0^2 (x^3 - 2x)\,dx = 4$$

5. Find the inflection points of $f(x) = x^4 - 4x^2$.

$$f''(x) = 12x^2 - 8, \text{ inflection points at } x = \pm\frac{2}{\sqrt{3}}$$

Hard Problems

1. Evaluate the integral $\int_0^\infty e^{-x^2}\,dx$ (Hint: Use the Gaussian integral formula).

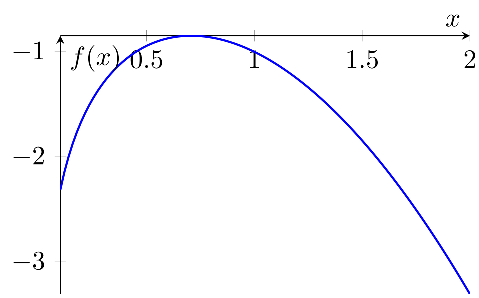


$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$

Follow-up Question: Evaluate the integral $\int_{-\infty}^\infty e^{-x^2}\,dx$ using the same technique.

$$\int_{-\infty}^\infty e^{-x^2}\,dx = \sqrt{\pi}$$

2. Find the maximum and minimum of $f(x) = \ln(x) - x^2$ on the interval $[0.1, 2]$.



$$\text{Max at } x = \frac{1}{2},\ \text{Min at } x = 2$$

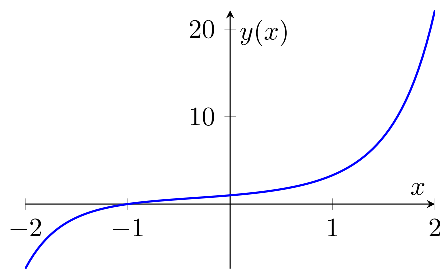
Follow-up Question (hard): Find the points of inflection of $f(x) = \ln(x) - x^2$.

$f''(x) = -\dfrac{1}{x^2} - 2 \Rightarrow$ No inflection points since the concavity does not change.

Follow-up Question (easy): What is the behavior of $f(x)$ as $x \to 0^+$ and as $x \to \infty$?

$$\lim_{x \to 0^+} f(x) = -\infty, \quad \lim_{x \to \infty} f(x) = -\infty$$

3. Find the solution to the differential equation $\frac{dy}{dx} = xy + 2x$.

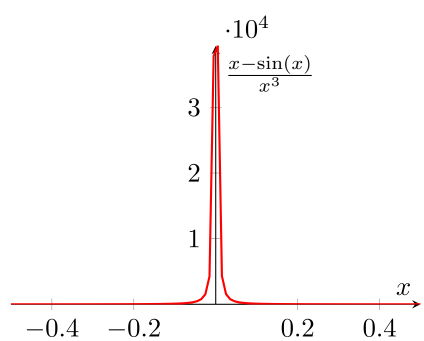


$$y = e^{x^2/2}(C + 2)$$

Follow-up Question (medium): Solve the same differential equation but with the initial condition $y(0) = 3$.

$$y = e^{x^2/2}(1 + 2)$$

4. Compute the limit $\lim_{x \to 0} \frac{x - \sin(x)}{x^3}$.



$$\lim_{x \to 0} \frac{x - \sin(x)}{x^3} = \frac{1}{6}$$

Follow-up Question (easy): Compute $\lim_{x\to 0} \frac{x-\tan(x)}{x^3}$.

$$\lim_{x\to 0} \frac{x-\tan(x)}{x^3} = \frac{1}{3}$$

5. Solve the following integral $\int_1^\infty \frac{\ln(x)}{x^2}\,dx$.

$$\int_1^\infty \frac{\ln(x)}{x^2}\,dx = -\frac{1}{2}$$

Follow-up Question (hard): Use integration by parts to evaluate $\int_1^\infty \frac{\ln(x)}{x^3}\,dx$.

$$\int_1^\infty \frac{\ln(x)}{x^3}\,dx = -\frac{1}{8}$$

Additional Questions

1. Prove that the function $f(x) = x^3 - 3x$ has exactly three real roots.

$$f(x) = x(x^2-3), \text{ roots are at } x = 0, \pm\sqrt{3}$$

2. Compute $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x$.

$$\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = e$$

3. Differentiate $f(x) = \frac{\sin(x)}{x^2+1}$.

$$f'(x) = \frac{x^2+1\cdot\cos(x) - 2x\cdot\sin(x)}{(x^2+1)^2}$$

4. Find the volume of the solid obtained by rotating the region bounded by $y = x^2$ and $y = 4$ around the x-axis.

$$V = \int_{-2}^{2} \pi(4^2 - x^4)\,dx = \frac{128\pi}{5}$$

5. Determine the radius of convergence of the series $\sum_{n=1}^\infty \frac{x^n}{n^2}$.

$$R = 1$$