Self Assignment: Multivariable Calculus: Partial Derivatives, Multiple Integrals

2024

Introduction

In this document, we will focus on two key topics in multivariable calculus:

- **Partial Derivatives**: Understanding how to differentiate functions of multiple variables.
- **Multiple Integrals**: Extending integration to functions of two or more variables.

Both concepts will be explained in a simple, step-by-step manner, with examples and illustrations where appropriate.

Partial Derivatives

Definition

A *partial derivative* measures how a function changes as one of its variables changes, while keeping the other variables fixed. This concept is crucial when dealing with functions that depend on more than one variable.

Consider a function $f(x, y)$ that depends on two variables, x and y. To find the partial derivative of f with respect to x, you observe how f changes as x changes, while keeping y constant. Similarly, for the partial derivative with respect to y, you observe how f changes as y changes, while keeping x constant.

Mathematical Notation

For a function $f(x, y)$, the partial derivative with respect to x is denoted by:

$$\frac{\partial f}{\partial x}$$

and is defined as:

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

This expression represents the limit of the average rate of change of f as Δx, the small change in x, approaches zero. It tells us how f changes as x changes slightly, while y remains fixed.

Similarly, the partial derivative with respect to y is denoted by:

$$\frac{\partial f}{\partial y}$$

and is defined as:

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

This represents the limit of the average rate of change of f as Δy, the small change in y, approaches zero. It measures how f changes as y changes slightly, while x is held constant.

Understanding with an Example

Let's take a closer look with an example. Suppose you have a function $f(x, y) = x^2y + y^3$. We want to find the partial derivatives with respect to x and y.

Step 1: Partial derivative with respect to x:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2y + y^3)$$

Here, we treat y as a constant. The derivative of x^2y with respect to x is $2xy$, and the derivative of y^3 with respect to x is 0 (since it's a constant with respect to x). Thus:

$$\frac{\partial f}{\partial x} = 2xy$$

Step 2: Partial derivative with respect to y:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2y + y^3)$$

Here, we treat x as a constant. The derivative of x^2y with respect to y is x^2, and the derivative of y^3 with respect to y is $3y^2$. Thus:

$$\frac{\partial f}{\partial y} = x^2 + 3y^2$$

Geometric Interpretation

To visualize partial derivatives, imagine a surface described by $f(x, y)$ over the xy-plane. A partial derivative gives you the slope of the surface in the direction of one variable.

- **Partial Derivative with Respect to x:** Imagine moving along the x-direction while keeping y fixed. The slope you encounter as you move in the x-direction is the partial derivative with respect to x.
- **Partial Derivative with Respect to y:** Similarly, if you move along the y-direction while keeping x fixed, the slope you encounter is the partial derivative with respect to y.

Illustration: (Include a 3D plot of $f(x, y) = x^2y + y^3$. Add vertical slices along the x-axis and y-axis to show how the function behaves as you change one variable while holding the other constant.)

In summary, partial derivatives are a powerful tool for analyzing how functions of multiple variables behave. By examining how a function changes with respect to each variable independently, you gain insights into the function's behavior in various directions.

Example

Consider the function $f(x, y) = x^2y + y^3$. Let's compute the partial derivatives with respect to x and y.

Step 1: Partial derivative with respect to x:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2y + y^3) = 2xy$$

Step 2: Partial derivative with respect to y:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2y + y^3) = x^2 + 3y^2$$

Geometric Interpretation

A partial derivative measures how the function changes as one variable changes while others remain fixed. It can be thought of as slicing through the graph of a function along a plane parallel to one of the coordinate axes.

Illustration: (Include a 3D graph of the function $f(x, y) = x^2y + y^3$, with slices along the x- and y-axes showing the partial derivatives.)

Multiple Integrals

Double Integrals

A *double integral* is a powerful tool in calculus used to compute the volume under a surface defined by a function of two variables. It extends the concept of a single integral to functions of two variables and is especially useful in applications involving areas and volumes.

Definition

For a function $f(x, y)$ defined over a region R in the xy-plane, the double integral of f over R is denoted by:

$$\int\int_R f(x, y)\, dA$$

Here, dA represents a small area element within the region R. Essentially, the double integral sums up the values of the function $f(x, y)$ over the entire region R, with each small area element contributing to the total volume.

Geometric Interpretation

Think of $f(x, y)$ as describing the height of a surface above each point (x, y) in the xy-plane. The double integral calculates the volume of the solid that lies under this surface and above the region R.

Imagine slicing the surface into thin slices parallel to the xy-plane. Each slice is a small rectangle (or element) with a height equal to $f(x, y)$. By summing the volumes of all these slices (small rectangles times their height), you obtain the total volume under the surface.

Example

Let's calculate the double integral of $f(x, y) = x^2 + y^2$ over the rectangular region R defined by $0 \leq x \leq 1$ and $0 \leq y \leq 1$.

Step 1: Set up the double integral:

$$\int_0^1 \int_0^1 (x^2 + y^2)\, dx\, dy$$

Step 2: Compute the inner integral (with respect to x) first:

$$\int_0^1 (x^2 + y^2)\, dx$$

Here, y^2 is treated as a constant while integrating with respect to x:

$$\int_0^1 x^2\, dx + \int_0^1 y^2\, dx$$

$$= \left[\frac{x^3}{3}\right]_0^1 + y^2\, [x]_0^1$$

$$= \frac{1}{3} + y^2$$

Step 3: Compute the outer integral (with respect to y):

$$\int_0^1 \left(\frac{1}{3} + y^2\right) dy$$

$$= \int_0^1 \frac{1}{3}\, dy + \int_0^1 y^2\, dy$$

$$= \frac{1}{3}\, [y]_0^1 + \left[\frac{y^3}{3}\right]_0^1$$

$$= \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

Thus, the value of the double integral is $\frac{2}{3}$.

Iterated Integrals

To compute a double integral, you generally perform integration in two steps. This method is called *iterated integration*, where you first integrate with respect to one variable (usually x), and then with respect to the other variable (usually y).

Step-by-Step Procedure:

1. Integrate the function with respect to the first variable while treating the other variable as a constant.
2. Integrate the result with respect to the second variable.

Illustration: (Include a 3D plot of $f(x, y) = x^2 + y^2$ over the region R. Show the surface above the xy-plane and the volume under this surface. Indicate the region R on the xy-plane.)

In summary, double integrals extend the concept of integration to functions of two variables, allowing you to compute areas and volumes by summing up contributions from small regions within a defined area. This approach is widely used in fields such as physics and engineering for solving problems involving multiple dimensions.

Example

Compute the double integral of $f(x, y) = x^2 + y^2$ over the region R, where $0 \leq x \leq 1$ and $0 \leq y \leq 1$.

Step 1: Set up the integral:

$$\int_0^1 \int_0^1 (x^2 + y^2)\, dx\, dy$$

Step 2: Compute the inner integral (with respect to x):

$$\int_0^1 (x^2 + y^2)\, dx = \int_0^1 x^2\, dx + \int_0^1 y^2\, dx = \frac{1}{3} + y^2$$

Step 3: Compute the outer integral (with respect to y):

$$\int_0^1 \left(\frac{1}{3} + y^2\right) dy = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

Thus, the value of the double integral is $\frac{2}{3}$.

Geometric Interpretation

Double integrals are a powerful tool for calculating the volume under a surface defined by a function of two variables. To understand this concept geometrically, consider the following:

Surface Definition: Suppose we have a function $f(x, y)$ that describes a surface in three-dimensional space. For example, let's consider the surface given by $z = x^2 + y^2$. This surface is a paraboloid that opens upwards.

Region of Integration: The double integral computes the volume under the surface $z = f(x, y)$ above a specific region R in the xy-plane. The region R is the area over which we integrate. For instance, if R is a rectangle defined by $0 \leq x \leq 1$ and $0 \leq y \leq 1$, the double integral will compute the volume under the surface over this rectangular region.

Volume Calculation: The double integral $\int \int_R f(x, y)\, dA$ essentially sums up the volume of infinitesimally small vertical prisms above each point in R. Each small prism has a height equal to the value of $f(x, y)$ at that point and a base area dA. Adding up all these small volumes gives the total volume under the surface.

In summary, double integrals provide a way to compute the total volume under a surface by summing up the contributions over a specified region in the plane. This geometric interpretation helps visualize how integration in multiple dimensions extends the concept of summing areas to summing volumes.

Conclusion

We have covered the basic concepts of partial derivatives and multiple integrals. Partial derivatives help us understand how functions change with respect to each variable independently, while multiple integrals extend the idea of integration to higher dimensions.