Introduction to Calculus

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Abstract

This document will showcase all work in Summarized form for Introduction to Calculus.

1 Introduction to Calculus

The following will be discussed:

- Increasing and decreasing functions
- piece-wise functions
- limits
- composite functions
- derivatives
- rates/velocity
- critical numbers
- MVT (Mean Value Theorem)
- tangent lines
- integrals
- optimization or integration
- definite and indefinite integrals
- absolute min and max
- first principles

Once you're familiar with the work, you can find various problems in the folder section of the University-Math Repository. A Detailed guide will be given. In the given Github, please visit the link University-Math, or head to Github and look for DylanPrinsloo.

2 Increasing and decreasing functions

2.1 Definitions

Increasing Functions

A function $f(x)$ is said to be **increasing** on an interval I if for all $x_1, x_2 \in I$ such that $x_1 < x_2$, we have:

$$f(x_1) \leq f(x_2).$$

If the inequality is strict, i.e. $f(x_1) < f(x_2)$, then $f(x)$ is said to be **strictly increasing** on the interval.

Decreasing Functions

A function $f(x)$ is said to be **decreasing** on an interval I if for all $x_1, x_2 \in I$ such that $x_1 < x_2$, we have:

$$f(x_1) \geq f(x_2).$$

If the inequality is strict, i.e. $f(x_1) > f(x_2)$, then $f(x)$ is said to be **strictly decreasing** on the interval.

First Derivative Test

To determine whether a function is increasing or decreasing on an interval, we can use its first derivative $f'(x)$.

Increasing

If $f'(x) > 0$ for all x in an interval, then $f(x)$ is **increasing** on that interval.

Decreasing

If $f'(x) < 0$ for all x in an interval, then $f(x)$ is **decreasing** on that interval.

Critical Points and the First Derivative

Critical points occur where $f'(x) = 0$ or where $f'(x)$ does not exist. These points can help identify changes in the function's behavior from increasing to decreasing (or vice versa).

Example

Consider the function $f(x) = x^3 - 3x^2 + 2x$.

- First, compute the derivative:
$$f'(x) = 3x^2 - 6x + 2.$$
- Set $f'(x) = 0$ to find the critical points:
$$3x^2 - 6x + 2 = 0.$$
Solving this quadratic equation gives the critical points $x = 1 \pm \frac{1}{\sqrt{3}}$.
- By analyzing the sign of $f'(x)$ on the intervals determined by the critical points, we can conclude that the function is increasing where $f'(x) > 0$ and decreasing where $f'(x) < 0$.

Piecewise Functions

A **piecewise function** is defined by different expressions depending on the value of the input x. The general form of a piecewise function is:

$$f(x) = \begin{cases} f_1(x), & \text{if } x \in I_1, \\ f_2(x), & \text{if } x \in I_2, \\ \vdots \\ f_n(x), & \text{if } x \in I_n. \end{cases}$$

where each $f_i(x)$ is the function that applies on the corresponding interval I_i.

Continuity of Piecewise Functions

For a piecewise function to be continuous at a point $x = c$, the following conditions must be satisfied:

- The limit $\lim_{x \to c^-} f(x)$ (from the left) exists,
- The limit $\lim_{x \to c^+} f(x)$ (from the right) exists,
- The function value at c, $f(c)$, exists,
- The left-hand limit, right-hand limit, and the function value at c are all equal:

$$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = f(c).$$

Example of a Piecewise Function

Consider the following piecewise function:

$$f(x) = \begin{cases} x^2, & \text{if } x < 0, \\ 2x + 1, & \text{if } 0 \leq x < 1, \\ \sin(\pi x), & \text{if } x \geq 1. \end{cases}$$

We can analyze this function for continuity:

- At $x = 0$:

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x^2 = 0, \quad \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (2x + 1) = 1.$$

  Since the left-hand limit and right-hand limit do not agree, $f(x)$ is discontinuous at $x = 0$.

- At $x = 1$:

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (2x + 1) = 3, \quad \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \sin(\pi x) = \sin(\pi) = 0.$$

  Thus, $f(x)$ is discontinuous at $x = 1$.

Derivatives of Piecewise Functions

To find the derivative of a piecewise function, differentiate each piece separately within its domain. At the points where the pieces meet (the boundary points of the intervals), check for differentiability by ensuring the left-hand derivative equals the right-hand derivative.

For example, consider the piecewise function:

$$g(x) = \begin{cases} x^3, & \text{if } x < 1, \\ 3x - 2, & \text{if } x \geq 1. \end{cases}$$

The derivative is:

$$g'(x) = \begin{cases} 3x^2, & \text{if } x < 1, \\ 3, & \text{if } x > 1. \end{cases}$$

At $x = 1$, the left-hand derivative is $g'(1^-) = 3 \cdot 1^2 = 3$, and the right-hand derivative is $g'(1^+) = 3$, so the function is differentiable at $x = 1$.

Summary of Limits for Piecewise Functions in Calculus

Definition of Limits

The **limit** of a function $f(x)$ as x approaches a value c is the value that $f(x)$ gets closer to as x gets closer to c. This is denoted as:

$$\lim_{x \to c} f(x) = L,$$

where L is the value the function approaches as $x \to c$.

For a piecewise function, limits must be evaluated separately on different intervals, especially at the points where the function changes its expression.

One-Sided Limits

For piecewise functions, it is important to consider the **left-hand limit** and the **right-hand limit**.

- The **left-hand limit** of $f(x)$ as $x \to c$ from the left (denoted $\lim_{x\to c^-} f(x)$) is the value that $f(x)$ approaches as x gets closer to c from values less than c. - The **right-hand limit** of $f(x)$ as $x \to c$ from the right (denoted $\lim_{x\to c^+} f(x)$) is the value that $f(x)$ approaches as x gets closer to c from values greater than c.

If both the left-hand limit and right-hand limit exist and are equal, then the two-sided limit exists and is equal to that common value. If they are not equal, the two-sided limit does not exist.

Example of a Piecewise Function with Limits

Consider the following piecewise function:

$$f(x) = \begin{cases} x^2 - 1, & \text{if } x < 1, \\ 2x + 3, & \text{if } x \geq 1. \end{cases}$$

Let's evaluate the limit as x approaches 1:

- **Left-hand limit**:
$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x^2 - 1) = 1^2 - 1 = 0.$$
- **Right-hand limit**:
$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (2x + 3) = 2(1) + 3 = 5.$$
- Since $\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$, the two-sided limit $\lim_{x\to 1} f(x)$ does not exist.

Continuity and Limits

A piecewise function is **continuous** at a point $x = c$ if:

1. The limit $\lim_{x\to c} f(x)$ exists,
2. The function value $f(c)$ exists,
3. The limit is equal to the function value:
$$\lim_{x\to c} f(x) = f(c).$$

In the example above, $f(x)$ is discontinuous at $x = 1$ because the two-sided limit does not exist.

Example with Continuous Piecewise Function

Consider another piecewise function:

$$g(x) = \begin{cases} 3x + 1, & \text{if } x \leq 2, \\ 7, & \text{if } x > 2. \end{cases}$$

Let's evaluate the limit as x approaches 2:

- **Left-hand limit**:
$$\lim_{x\to 2^-} g(x) = \lim_{x\to 2^-} (3x + 1) = 3(2) + 1 = 7.$$
- **Right-hand limit**:
$$\lim_{x\to 2^+} g(x) = 7.$$
- Since $\lim_{x\to 2^-} g(x) = \lim_{x\to 2^+} g(x) = 7$, the two-sided limit exists, and $\lim_{x\to 2} g(x) = 7$.

Moreover, $g(2) = 7$, so the function is continuous at $x = 2$.
Summary of Limits for Composite Functions in Calculus

Definition of Composite Functions

A **composite function** is a function created by applying one function to the result of another. If $f(x)$ and $g(x)$ are two functions, then the composite function $(f \circ g)(x)$ is defined as:

$$(f \circ g)(x) = f(g(x)).$$

This means we first apply $g(x)$, then apply f to the result.

Limits of Composite Functions

To evaluate the limit of a composite function $\lim_{x \to c} f(g(x))$, it often helps to use the **limit laws** and the **continuity** of the outer function f.

Case 1: $g(x)$ Approaches a Finite Limit

If $\lim_{x \to c} g(x) = L$ and f is continuous at L, then:

$$\lim_{x \to c} f(g(x)) = f\left(\lim_{x \to c} g(x)\right) = f(L).$$

Case 2: $g(x)$ Approaches $\pm\infty$

If $\lim_{x \to c} g(x) = \pm\infty$ and $\lim_{x \to \pm\infty} f(x)$ exists, then:

$$\lim_{x \to c} f(g(x)) = \lim_{x \to \pm\infty} f(x).$$

Summary of Limits Involving Infinity

Limits Involving Infinity

Limits involving infinity are used to describe the behavior of functions as they approach infinitely large or small values. They are crucial for understanding asymptotic behavior, continuity at infinity, and vertical asymptotes.

Limits as $x \to \infty$ or $x \to -\infty$

To find the limit of a function as x approaches infinity (∞) or negative infinity ($-\infty$), follow these principles:

1. **Rational Functions**: For a rational function $f(x) = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials:

- If the degree of $P(x)$ is less than the degree of $Q(x)$, then:

$$\lim_{x \to \pm\infty} \frac{P(x)}{Q(x)} = 0.$$

- If the degree of $P(x)$ is equal to the degree of $Q(x)$, then:

$$\lim_{x \to \pm\infty} \frac{P(x)}{Q(x)} = \frac{\text{leading coefficient of } P(x)}{\text{leading coefficient of } Q(x)}.$$

- If the degree of $P(x)$ is greater than the degree of $Q(x)$, then:

$$\lim_{x \to \pm\infty} \frac{P(x)}{Q(x)} = \pm\infty.$$

2. **Exponential Functions**:

- For $f(x) = e^x$:

$$\lim_{x\to\infty} e^x = \infty,$$

$$\lim_{x\to-\infty} e^x = 0.$$

- For $f(x) = e^{-x}$:

$$\lim_{x\to\infty} e^{-x} = 0,$$

$$\lim_{x\to-\infty} e^{-x} = \infty.$$

3. **Logarithmic Functions**:

- For $f(x) = \ln(x)$:

$$\lim_{x\to\infty} \ln(x) = \infty,$$

$$\lim_{x\to 0^+} \ln(x) = -\infty.$$

Limits as $x \to a$ and $f(x) \to \pm\infty$

When finding limits where $f(x)$ approaches $\pm\infty$ as x approaches a finite value a, we are dealing with vertical asymptotes.

- If $\lim_{x\to a} f(x) = \infty$, the function $f(x)$ grows without bound as x approaches a from either side.
- If $\lim_{x\to a} f(x) = -\infty$, the function $f(x)$ decreases without bound as x approaches a from either side.

Example: Limits Involving Infinity

1. Find:

$$\lim_{x\to\infty} \frac{2x^3 - x + 1}{x^3 + 4}.$$

Since the degrees of the numerator and denominator are the same (both 3), the limit is:

$$\lim_{x\to\infty} \frac{2x^3 - x + 1}{x^3 + 4} = \frac{2}{1} = 2.$$

2. Find:

$$\lim_{x\to 0^+} \ln(x).$$

Since $\ln(x) \to -\infty$ as $x \to 0^+$:

$$\lim_{x\to 0^+} \ln(x) = -\infty.$$

3. Find:

$$\lim_{x\to 2} \frac{1}{x-2}.$$

As $x \to 2$, the denominator approaches 0. The behavior depends on the direction:

$$\lim_{x\to 2^+} \frac{1}{x-2} = +\infty,$$

$$\lim_{x\to 2^-} \frac{1}{x-2} = -\infty.$$

Example of Composite Function with Limits

Consider the composite function:

$$f(x) = \sqrt{1 + \sin(2x)}.$$

To evaluate $\lim_{x\to\pi} f(x)$, proceed as follows:

- Start by evaluating the inner function:

$$g(x) = \sin(2x).$$

$$\lim_{x\to\pi} g(x) = \sin(2\pi) = 0.$$

- Now apply the outer function:

$$\lim_{x\to\pi} f(x) = \sqrt{1 + \lim_{x\to\pi} \sin(2x)} = \sqrt{1+0} = 1.$$

Thus, $\lim_{x\to\pi} \sqrt{1+\sin(2x)} = 1$.

Squeeze Theorem for Composite Functions

Sometimes, it is difficult to directly evaluate the limit of a composite function. In such cases, the **squeeze theorem** can be useful. It states that if $h(x) \leq f(x) \leq k(x)$ for all x near c (except possibly at c), and if:

$$\lim_{x\to c} h(x) = \lim_{x\to c} k(x) = L,$$

then:

$$\lim_{x\to c} f(x) = L.$$

Example Using Squeeze Theorem

Consider the limit:

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right).$$

Here, the inner function is $g(x) = \sin\left(\frac{1}{x}\right)$, and we know that $-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$ for all x.

Thus, we have:

$$-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2.$$

Taking the limit of the outer functions as $x \to 0$:

$$\lim_{x\to 0} -x^2 = 0 \quad \text{and} \quad \lim_{x\to 0} x^2 = 0.$$

By the squeeze theorem:

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0.$$

Limits and Continuity in Composite Functions

If the outer function f is continuous at the point where the inner function $g(x)$ converges, then the limit of the composite function $f(g(x))$ is simply f applied to the limit of $g(x)$. However, if f is not continuous at that point, special care is needed to determine whether the limit exists.

Summary of Derivatives, Product Rule, Chain Rule, and Quotient Rule in Calculus

Derivatives

The **derivative** of a function measures the rate at which the function value changes with respect to a change in its input value. The derivative of a function $f(x)$ with respect to x is defined as:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x},$$

provided the limit exists. It represents the slope of the tangent line to the curve $y = f(x)$ at any point x.

Basic Derivative Rules

Some common derivative rules include:

- **Power Rule**: If $f(x) = x^n$, then $f'(x) = nx^{n-1}$.
- **Constant Rule**: If $f(x) = c$ (where c is a constant), then $f'(x) = 0$.
- **Sum/Difference Rule**: If $f(x) = g(x) + h(x)$, then $f'(x) = g'(x) + h'(x)$.

Product Rule

The **product rule** is used to differentiate the product of two functions. If $f(x) = g(x) \cdot h(x)$, then the derivative of $f(x)$ is:

$$f'(x) = g'(x)h(x) + g(x)h'(x).$$

Example of Product Rule

Let $g(x) = x^2$ and $h(x) = \sin(x)$. To find $\frac{d}{dx}[x^2 \cdot \sin(x)]$:

$$\begin{aligned} f'(x) &= \frac{d}{dx}[x^2] \cdot \sin(x) + x^2 \cdot \frac{d}{dx}[\sin(x)] \\ &= 2x \cdot \sin(x) + x^2 \cdot \cos(x). \end{aligned}$$

Quotient Rule

The **quotient rule** is used to differentiate the quotient of two functions. If $f(x) = \frac{g(x)}{h(x)}$, then the derivative of $f(x)$ is:

$$f'(x) = \frac{g'(x)h(x) - g(x)h'(x)}{[h(x)]^2}.$$

Example of Quotient Rule

Let $g(x) = x^3$ and $h(x) = x + 1$. To find $\frac{d}{dx}\left[\frac{x^3}{x+1}\right]$:

$$\begin{aligned} f'(x) &= \frac{\frac{d}{dx}[x^3] \cdot (x+1) - x^3 \cdot \frac{d}{dx}[x+1]}{(x+1)^2} \\ &= \frac{3x^2 \cdot (x+1) - x^3 \cdot 1}{(x+1)^2} \\ &= \frac{3x^3 + 3x^2 - x^3}{(x+1)^2} \\ &= \frac{2x^3 + 3x^2}{(x+1)^2}. \end{aligned}$$

Chain Rule

The **chain rule** is used to differentiate a composite function. If $f(x) = g(h(x))$, then the derivative of $f(x)$ is:

$$f'(x) = g'(h(x)) \cdot h'(x).$$

Example of Chain Rule

Let $f(x) = \sin(x^2)$. To find $\frac{d}{dx}[\sin(x^2)]$:

$$\begin{aligned} f'(x) &= \cos(x^2) \cdot \frac{d}{dx}[x^2] \\ &= \cos(x^2) \cdot 2x. \end{aligned}$$

Combined Example

Let $f(x) = \frac{\sin(x^2)}{x^3+1}$. Using the quotient rule and chain rule, we find $f'(x)$:

$$f'(x) = \frac{\cos(x^2) \cdot 2x \cdot (x^3+1) - \sin(x^2) \cdot 3x^2}{(x^3+1)^2}.$$

Summary of Rates of Change and Velocity in Calculus

Rates of Change

The **rate of change** of a function describes how one quantity changes in relation to another. It is commonly used to measure how a function $f(x)$ changes as its input x changes. The average rate of change of $f(x)$ over an interval $[a, b]$ is given by:

$$\text{Average rate of change} = \frac{f(b) - f(a)}{b - a}.$$

The **instantaneous rate of change** is the derivative of the function at a particular point, which represents the slope of the tangent line to the curve at that point. It is given by:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

This derivative tells us how $f(x)$ is changing at the exact point x.

Velocity

In the context of motion, **velocity** is the rate of change of position with respect to time. It is the derivative of the position function $s(t)$ with respect to time t. If $s(t)$ represents the position of an object as a function of time, then the velocity $v(t)$ is:

$$v(t) = \frac{ds}{dt} = s'(t).$$

The velocity represents both the speed and direction of motion.

Example of Velocity

Suppose the position of a particle is given by:

$$s(t) = 4t^2 - 3t + 2.$$

To find the velocity at time t, we differentiate $s(t)$ with respect to t:

$$v(t) = \frac{d}{dt}[4t^2 - 3t + 2] = 8t - 3.$$

Thus, the velocity at any time t is $v(t) = 8t - 3$.

Acceleration

Acceleration is the rate of change of velocity with respect to time. It is the derivative of the velocity function $v(t)$ with respect to time:

$$a(t) = \frac{dv}{dt} = v'(t) = s''(t).$$

Acceleration tells us how the velocity is changing over time.

Example of Acceleration

For the position function $s(t) = 4t^2 - 3t + 2$, we already found that the velocity is $v(t) = 8t - 3$. The acceleration is:

$$a(t) = \frac{d}{dt}[8t - 3] = 8.$$

Thus, the acceleration is constant at 8 units per time squared.

Applications of Rates of Change and Velocity

Related Rates

In many real-world problems, two or more quantities are related and change with respect to time. These problems are known as **related rates** problems. To solve related rates problems, we differentiate both sides of an equation with respect to time t, applying the chain rule as necessary.

Example of Related Rates

Consider a balloon being inflated so that its volume increases at a rate of 100 cubic centimeters per second. If the volume V of a sphere is given by:

$$V = \frac{4}{3}\pi r^3,$$

we are asked to find the rate at which the radius r is increasing when the radius is 5 cm.

First, differentiate both sides of the volume formula with respect to time t:

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

We know $\frac{dV}{dt} = 100 \text{ cm}^3/\text{s}$ and $r = 5$ cm. Substituting these values into the equation:

$$100 = 4\pi(5)^2 \frac{dr}{dt}.$$

Solving for $\frac{dr}{dt}$:

$$\frac{dr}{dt} = \frac{100}{4\pi(25)} = \frac{1}{\pi} \text{ cm/s}.$$

Thus, the radius is increasing at a rate of $\frac{1}{\pi}$ cm/s.

Velocity and Speed

While velocity is a vector quantity (meaning it has both magnitude and direction), **speed** is the magnitude of velocity and is always non-negative. If $v(t)$ is the velocity, then the speed $|v(t)|$ is:

$$\text{Speed} = |v(t)|.$$

Example of Speed

For the velocity function $v(t) = 8t - 3$, the speed at time t is:

$$\text{Speed} = |8t - 3|.$$

At $t = 1$, the speed is:

$$\text{Speed} = |8(1) - 3| = |8 - 3| = 5.$$

Summary of Critical Numbers in Calculus

Critical Numbers

In calculus, **critical numbers** (or **critical points**) of a function $f(x)$ are the points where the derivative of the function is zero or undefined. These points are important because they help identify local maxima, minima, or points of inflection.

Definition of Critical Numbers

A critical number c of a function $f(x)$ occurs where:

- $f'(c) = 0$ (the derivative is zero), or
- $f'(c)$ does not exist (the derivative is undefined).

For c to be a critical number, it must also be within the domain of the function $f(x)$.

Finding Critical Numbers

To find the critical numbers of a function $f(x)$:

1. Take the derivative of the function, $f'(x)$.
2. Solve $f'(x) = 0$ to find potential critical points.
3. Identify any points where $f'(x)$ is undefined but where x is in the domain of $f(x)$.

Example of Finding Critical Numbers

Consider the function $f(x) = x^3 - 3x^2 + 4$. To find the critical numbers:

1. Take the derivative of $f(x)$:
$$f'(x) = 3x^2 - 6x.$$
2. Solve $f'(x) = 0$:
$$3x^2 - 6x = 0 \implies 3x(x - 2) = 0.$$
The solutions are $x = 0$ and $x = 2$.
3. Since the derivative is defined for all x, there are no points where $f'(x)$ is undefined.
Thus, the critical numbers are $x = 0$ and $x = 2$.

Classifying Critical Numbers

After finding the critical numbers, the next step is often to classify them as local maxima, minima, or points of inflection. This can be done using:

- **First Derivative Test**: Analyze the sign changes of $f'(x)$ around the critical numbers to determine if the function is increasing or decreasing.
- **Second Derivative Test**: If $f''(x)$ exists, evaluate the second derivative at the critical numbers. If $f''(c) > 0$, then c is a local minimum; if $f''(c) < 0$, then c is a local maximum.

Example of Classifying Critical Numbers

For the function $f(x) = x^3 - 3x^2 + 4$, we found critical numbers $x = 0$ and $x = 2$. To classify these points:

- Take the second derivative:
$$f''(x) = 6x - 6.$$
- Evaluate the second derivative at the critical numbers:
$$f''(0) = 6(0) - 6 = -6 \quad \text{(local maximum)},$$
$$f''(2) = 6(2) - 6 = 6 \quad \text{(local minimum)}.$$

Thus, $x = 0$ is a local maximum, and $x = 2$ is a local minimum.

Conclusion

Critical numbers are essential in analyzing the behavior of functions. They help in locating potential turning points, which can be local maxima, minima, or inflection points. Once the critical numbers are found, the first and second derivative tests can be used to classify these points and better understand the function's graph.

Summary of the Mean Value Theorem in Calculus

Mean Value Theorem

The **Mean Value Theorem (MVT)** is a fundamental result in calculus that provides a relationship between the derivative of a function and the function's average rate of change over an interval. The theorem states that for a function $f(x)$ that satisfies certain conditions, there exists at least one point within the interval where the instantaneous rate of change (derivative) is equal to the average rate of change.

Statement of the Mean Value Theorem

Let $f(x)$ be a function that satisfies the following conditions:

1. $f(x)$ is continuous on the closed interval $[a, b]$,
2. $f(x)$ is differentiable on the open interval (a, b).

Then there exists at least one point $c \in (a, b)$ such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Interpretation of the Mean Value Theorem

The Mean Value Theorem states that there is at least one point c in the interval (a, b) where the instantaneous rate of change (the derivative) of the function $f(x)$ is equal to the average rate of change of the function over the interval $[a, b]$.

In other words, at some point c, the slope of the tangent line to the graph of $f(x)$ is the same as the slope of the secant line connecting the points $(a, f(a))$ and $(b, f(b))$.

Geometrical Interpretation

Geometrically, the Mean Value Theorem guarantees that for a continuous and differentiable function, there is a point on the graph where the tangent line is parallel to the secant line that passes through the endpoints of the interval. This is illustrated as:

$$\text{Slope of the tangent line at } c = \text{Slope of the secant line}.$$

Example of the Mean Value Theorem

Consider the function $f(x) = x^2$ on the interval $[1, 4]$. The function is continuous on $[1, 4]$ and differentiable on $(1, 4)$.

The average rate of change of $f(x)$ over $[1, 4]$ is:

$$\frac{f(4) - f(1)}{4 - 1} = \frac{16 - 1}{3} = \frac{15}{3} = 5.$$

To apply the Mean Value Theorem, we find a point $c \in (1, 4)$ such that the derivative $f'(c)$ is equal to 5:

$$f'(x) = 2x.$$

We solve for c by setting $f'(c) = 5$:

$$2c = 5 \quad \Rightarrow \quad c = \frac{5}{2}.$$

Thus, there exists a point $c = \frac{5}{2}$ in the interval $(1, 4)$ where the instantaneous rate of change of the function is equal to the average rate of change.

Rolle's Theorem as a Special Case

Rolle's Theorem is a special case of the Mean Value Theorem. If a function satisfies the conditions of the Mean Value Theorem and, in addition, $f(a) = f(b)$, then there exists a point $c \in (a, b)$ where the derivative is zero:

$$f'(c) = 0.$$

Example of Rolle's Theorem

Consider the function $f(x) = x^2 - 4x + 3$ on the interval $[1, 3]$. The function is continuous on $[1, 3]$ and differentiable on $(1, 3)$, and $f(1) = f(3) = 0$. By Rolle's Theorem, there exists a point $c \in (1, 3)$ such that $f'(c) = 0$.

The derivative of $f(x)$ is:

$$f'(x) = 2x - 4.$$

Setting $f'(c) = 0$, we solve for c:

$$2c - 4 = 0 \quad \Rightarrow \quad c = 2.$$

Thus, $f'(2) = 0$, confirming the result of Rolle's Theorem.

Summary of Tangent Lines in Calculus

Tangent Lines

In calculus, a **tangent line** to a curve at a given point is a straight line that touches the curve at that point without crossing it, and its slope represents the instantaneous rate of change of the function at that point. The equation of the tangent line can be derived using the derivative of the function.

Equation of the Tangent Line

Given a function $f(x)$ that is differentiable at a point $x = a$, the equation of the tangent line to the graph of $f(x)$ at the point $(a, f(a))$ is:

$$y - f(a) = f'(a)(x - a),$$

where:

- $f'(a)$ is the slope of the tangent line (the derivative of $f(x)$ at $x = a$),
- $(a, f(a))$ is the point of tangency.

Slope of the Tangent Line

The slope of the tangent line at the point $(a, f(a))$ is given by the derivative of the function at that point:

$$\text{Slope} = f'(a).$$

The derivative represents the rate of change of the function at $x = a$, so the slope of the tangent line is the same as the instantaneous rate of change of the function at that point.

Example of Finding the Tangent Line

Consider the function $f(x) = x^2$, and find the equation of the tangent line at the point where $x = 1$.

1. First, evaluate $f(1)$:

$$f(1) = 1^2 = 1.$$

   The point of tangency is $(1, 1)$.

2. Next, find the derivative of $f(x) = x^2$:

$$f'(x) = 2x.$$

   At $x = 1$, the slope of the tangent line is:

$$f'(1) = 2(1) = 2.$$

3. Using the point-slope form of the equation of the tangent line:

$$y - 1 = 2(x - 1).$$

   Simplifying this equation gives:

$$y = 2x - 1.$$

Thus, the equation of the tangent line to the graph of $f(x) = x^2$ at $x = 1$ is $y = 2x - 1$.

Tangent Line Approximation

The tangent line can also be used to approximate the value of a function near the point of tangency. For small changes in x around a, the value of the function $f(x)$ can be approximated by the value of the tangent line:

$$f(x) \approx f(a) + f'(a)(x - a).$$

This approximation is known as the **linear approximation** or **tangent line approximation**.

Example of Tangent Line Approximation

For the function $f(x) = \sqrt{x}$, approximate $f(4.1)$ using the tangent line at $x = 4$.

1. The point of tangency is $(4, f(4)) = (4, 2)$.

2. The derivative of $f(x) = \sqrt{x}$ is:

$$f'(x) = \frac{1}{2\sqrt{x}}.$$

   At $x = 4$, the slope of the tangent line is:

$$f'(4) = \frac{1}{2 \times 2} = \frac{1}{4}.$$

3. The equation of the tangent line is:

$$y - 2 = \frac{1}{4}(x - 4).$$

   Simplifying gives:

$$y = \frac{1}{4}x + 1.$$

4. To approximate $f(4.1)$, substitute $x = 4.1$ into the tangent line equation:

$$y = \frac{1}{4}(4.1) + 1 = 1.025 + 1 = 2.025.$$

Thus, the tangent line approximation of $f(4.1)$ is 2.025.

Conclusion

Tangent lines provide a powerful tool for analyzing the behavior of functions at specific points, as well as for approximating the values of functions near those points. By using the derivative, we can determine the slope of the tangent line and derive its equation, which can be used for various applications, including approximation.

Summary of Integrals in Calculus

Integrals

In calculus, an **integral** represents the accumulation of quantities and is often interpreted as the area under a curve. There are two main types of integrals: definite and indefinite integrals.

Indefinite Integrals

An **indefinite integral** (or antiderivative) represents a family of functions whose derivative is the given function. It is written as:

$$\int f(x)\,dx.$$

The result is the antiderivative of $f(x)$ plus a constant C, because differentiation eliminates constants:

$$\int f(x)\,dx = F(x) + C,$$

where $F'(x) = f(x)$.

Basic Properties of Indefinite Integrals

- Linearity:
$$\int [af(x) + bg(x)]\,dx = a\int f(x)\,dx + b\int g(x)\,dx.$$
- Constant Rule:
$$\int c\,dx = cx + C.$$
- Power Rule:
$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1.$$

Example of Indefinite Integral

Find the indefinite integral of $f(x) = 3x^2 - 4x + 5$:

$$\int (3x^2 - 4x + 5)\,dx = x^3 - 2x^2 + 5x + C.$$

Definite Integrals

A **definite integral** calculates the exact accumulation of a quantity over an interval $[a, b]$ and is written as:

$$\int_a^b f(x)\,dx.$$

It gives the net area under the curve $y = f(x)$ from $x = a$ to $x = b$.

Fundamental Theorem of Calculus

The **Fundamental Theorem of Calculus** connects differentiation and integration and consists of two parts:

1. If $F(x)$ is the antiderivative of $f(x)$, then:

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

2. If $f(x)$ is continuous on $[a, b]$, then the function:

$$F(x) = \int_a^x f(t)\,dt$$

is differentiable and $F'(x) = f(x)$.

Example of Definite Integral

Evaluate $\int_1^3 (2x + 1)\,dx$:

$$\int_1^3 (2x + 1)\,dx = \left[x^2 + x\right]_1^3 = (9 + 3) - (1 + 1) = 10.$$

Applications of Definite Integrals

Definite integrals have several applications in physics, economics, and engineering. Some key applications include:

- **Area under a curve**: The integral of $f(x)$ from a to b gives the area under the curve $y = f(x)$ between the points $x = a$ and $x = b$.
- **Accumulation of quantities**: For example, integrating a rate of change gives the total change.
- **Volume of solids of revolution**: Using techniques like the *disk* or *shell* method, integrals can compute the volume of objects generated by rotating curves around axes.
- **Work**: In physics, the integral of force over distance gives the work done.

Properties of Definite Integrals

- Linearity:

$$\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

- Additivity:

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx.$$

- Reversal of Limits:

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

Example of Definite Integral with Application

Find the area under the curve $y = x^2$ from $x = 0$ to $x = 2$:

$$\int_0^2 x^2\,dx = \left[\frac{x^3}{3}\right]_0^2 = \frac{8}{3} - 0 = \frac{8}{3}.$$

Thus, the area under the curve is $\frac{8}{3}$ square units.

Summary of Optimization and Integration in Calculus

Optimization

Optimization involves finding the maximum or minimum values of a function. In calculus, this is typically done using derivatives to locate critical points where these extrema occur.

Steps for Optimization

1. **Identify the Objective Function**: Determine the function $f(x)$ that needs to be optimized.

2. **Find the Derivative**: Compute the derivative $f'(x)$.

3. **Find Critical Points**: Solve $f'(x) = 0$ to find critical points where the function may have local extrema.

4. **Check for Critical Points**: Verify if these critical points are within the domain of the function and check if they represent maxima or minima.

5. **Second Derivative Test**:

- Compute the second derivative $f''(x)$.
- If $f''(x) > 0$ at a critical point, $f(x)$ has a local minimum at that point.
- If $f''(x) < 0$ at a critical point, $f(x)$ has a local maximum at that point.
- If $f''(x) = 0$, the test is inconclusive, and further analysis is needed.

6. **Evaluate Endpoints**: For optimization problems over a closed interval, also evaluate the function at the endpoints.

Example of Optimization

Consider $f(x) = -2x^2 + 4x + 1$.

1. Find the derivative:

$$f'(x) = -4x + 4.$$

2. Set $f'(x) = 0$:

$$-4x + 4 = 0 \quad \Rightarrow \quad x = 1.$$

3. Compute the second derivative:

$$f''(x) = -4.$$

Since $f''(x) < 0$, $x = 1$ is a local maximum.

4. Evaluate the function at $x = 1$:

$$f(1) = -2(1)^2 + 4(1) + 1 = 3.$$

Thus, the local maximum value is 3 at $x = 1$.

Integration

Integration is a fundamental concept in calculus that represents the accumulation of quantities and can be used to find areas, volumes, and other quantities.

Indefinite Integrals

An **indefinite integral** represents a family of functions whose derivative is the integrand:

$$\int f(x)\,dx = F(x) + C,$$

where $F'(x) = f(x)$ and C is the constant of integration.

Definite Integrals

A **definite integral** computes the total accumulation of a quantity over a specified interval:

$$\int_a^b f(x)\,dx.$$

It provides the net area under the curve from $x = a$ to $x = b$.

Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus connects differentiation and integration:

1. **Part 1**: If $F(x)$ is the antiderivative of $f(x)$, then:

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

2. **Part 2**: If $f(x)$ is continuous on $[a, b]$, then:

$$F(x) = \int_a^x f(t)\,dt$$

is differentiable and $F'(x) = f(x)$.

Applications of Integration

Integration has various applications including:

- **Area under a Curve**: The integral of $f(x)$ from a to b gives the area under $y = f(x)$ between $x = a$ and $x = b$.
- **Volume of Solids of Revolution**: Using methods such as the disk or shell method.
- **Work and Energy**: In physics, the integral of force over distance gives work.
- **Probability**: Integration is used in probability to find expected values and distributions.

Example of Application

Find the area under the curve $y = x^3$ from $x = 1$ to $x = 3$:

$$\int_1^3 x^3\,dx = \left[\frac{x^4}{4}\right]_1^3 = \frac{3^4}{4} - \frac{1^4}{4} = \frac{81}{4} - \frac{1}{4} = 20.$$

Thus, the area under the curve is 20 square units.
Summary of Absolute Maximum and Minimum

Absolute Maximum and Minimum

In calculus, finding the **absolute maximum** and **absolute minimum** of a function involves determining the highest and lowest values of the function over its entire domain or a specified interval. These values are also known as global extrema.

Absolute Maximum

The **absolute maximum** of a function $f(x)$ on a domain D is the largest value that $f(x)$ attains within D. Formally, $f(x)$ has an absolute maximum at $x = c$ if:

$$f(c) \geq f(x) \quad \text{for all } x \in D.$$

Absolute Minimum

The **absolute minimum** of a function $f(x)$ on a domain D is the smallest value that $f(x)$ attains within D. Formally, $f(x)$ has an absolute minimum at $x = c$ if:

$$f(c) \leq f(x) \quad \text{for all } x \in D.$$

Finding Absolute Extrema on a Closed Interval

To find the absolute maximum and minimum of a function $f(x)$ on a closed interval $[a, b]$, follow these steps:

1. **Find the Critical Points**: - Compute the derivative $f'(x)$. - Solve $f'(x) = 0$ to find critical points within (a, b).

2. **Evaluate the Function at Critical Points**: - Compute $f(x)$ at each critical point found in step 1.

3. **Evaluate the Function at Endpoints**: - Compute $f(a)$ and $f(b)$.

4. **Compare Values**: - Compare the values obtained in steps 2 and 3. - The largest value is the absolute maximum, and the smallest value is the absolute minimum.

Example

Find the absolute maximum and minimum of $f(x) = x^3 - 3x^2 + 4$ on the interval $[0, 3]$.

1. Compute the derivative:
$$f'(x) = 3x^2 - 6x.$$

2. Find the critical points by setting the derivative to zero:
$$3x^2 - 6x = 0 \quad \Rightarrow \quad 3x(x-2) = 0 \quad \Rightarrow \quad x = 0 \text{ or } x = 2.$$

3. Evaluate $f(x)$ at the critical points and endpoints:
$$f(0) = 0^3 - 3 \cdot 0^2 + 4 = 4,$$
$$f(2) = 2^3 - 3 \cdot 2^2 + 4 = 8 - 12 + 4 = 0,$$
$$f(3) = 3^3 - 3 \cdot 3^2 + 4 = 27 - 27 + 4 = 4.$$

4. Compare the values:
$$f(0) = 4, \quad f(2) = 0, \quad f(3) = 4.$$

   Thus, the absolute maximum is 4 (attained at $x = 0$ and $x = 3$), and the absolute minimum is 0 (attained at $x = 2$).

Notes

- The absolute extrema may occur at critical points or at the endpoints of the interval.
- For functions defined on an open interval, the absolute extrema may not exist within the interval but can be approached asymptotically.

Summary of Derivatives from First Principles

Derivatives from First Principles

The derivative of a function $f(x)$ at a point $x = a$ can be defined using the concept of limits and is known as the **definition of the derivative** or **first principles** of differentiation.

Definition of the Derivative

The derivative of $f(x)$ at $x = a$ is defined as:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

This limit, if it exists, gives the slope of the tangent line to the curve at $x = a$ and represents the instantaneous rate of change of the function at that point.

Steps to Compute the Derivative from First Principles

1. **Set Up the Difference Quotient**: Compute the expression:

$$\frac{f(a+h) - f(a)}{h}.$$

2. **Take the Limit as $h \to 0$**: Evaluate:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Example

Find the derivative of $f(x) = x^2$ at $x = 3$ using first principles.

1. Compute the difference quotient:

$$\frac{f(3+h) - f(3)}{h} = \frac{(3+h)^2 - 3^2}{h}.$$

2. Simplify the expression:

$$(3+h)^2 = 9 + 6h + h^2,$$
$$f(3+h) - f(3) = (9 + 6h + h^2) - 9 = 6h + h^2,$$
$$\frac{f(3+h) - f(3)}{h} = \frac{6h + h^2}{h} = 6 + h.$$

3. Take the limit as $h \to 0$:

$$f'(3) = \lim_{h \to 0} (6 + h) = 6.$$

Thus, the derivative of $f(x) = x^2$ at $x = 3$ is 6.

General Formula for a Polynomial Function

For a general polynomial function $f(x) = x^n$, the derivative at $x = a$ can be computed using first principles as follows:

$$f'(a) = \lim_{h \to 0} \frac{(a+h)^n - a^n}{h}.$$

Using the Binomial Theorem, this simplifies to:

$$f'(a) = \lim_{h \to 0} \frac{a^n + na^{n-1}h + \text{higher-order terms} - a^n}{h} = na^{n-1}.$$

Thus, the derivative of $f(x) = x^n$ is:

$$f'(x) = nx^{n-1}.$$

Conclusion

The method of first principles provides a foundational understanding of differentiation, showing how derivatives represent the limit of the average rate of change as the interval approaches zero. This approach is fundamental for understanding the concept of the derivative and for deriving formulas for derivatives of various functions.